# On the Dynamics of Manufacturing Systems
# A State Space Perspective

B. Scholz-Reiter, M. Freitag

Department of Planning and Control of Production Systems

University of Bremen, Bremen, Germany

**Abstract**

Manufacturing systems are considered to be dynamical systems in terms of Dynamical Systems Theory. A dynamical model of a flow line is introduced and analysed by using its state space. The work systems of the flow line are compared to a pendulum oscillator, which are driven by the part input and damped by their processing capacity. Depending on the ratio between part input and capacity, the underloaded system corresponds to a damped pendulum, whereas the overloaded system behaves like a driven pendulum. The balanced manufacturing system corresponds to an ideal pendulum, where driving and damping forces are equal and the dynamics are generally periodic. Finally, a relation between system dynamics and system performance is presented by using the system variable 'work-in-process'.

**Keywords**:
Manufacturing System, Dynamical Systems Theory, Dynamical Model, State Space

## 1 INTRODUCTION

Manufacturing systems are traditionally described by stochastic models that are mostly realised by queuing networks or event-discrete simulation. Within a queuing model, the machines and their input buffers are represented by servers with queues in front of it. The discrete part flow through the queuing network is described by formulas of stochastic queuing theory. Such models are good tools for analysing the overall system performance under different boundary conditions and control policies.

But a larger number of machines, buffers and processed parts lead to complex queuing networks which are hard to analyse using the formulas of queuing theory. In that case, it is appropriate to use the discrete-event simulation for analysing the system. The simulation provides results about stability and performance of the system, plus the chance to analyse its dynamic behaviour. Beaumariage et al. analysed a manufacturing system via simulation and showed the sensitive dependence of throughput times and output patterns on initial conditions and control policies [1]. They state the reentrant structure of their model and a high work load as being the origin of this unstable dynamic behaviour. Such complex dynamics (and possibly deterministic chaos) make it difficult to predict or control the performance of manufacturing systems.

To understand and describe such complex dynamic behaviour, manufacturing systems are considered to be dynamical systems in terms of Dynamical Systems Theory. This theory provides concepts and methods for modelling and analysing the dynamics of manufacturing systems. The objectives of this approach are a classification of the system dynamics to decide whether the system is periodic, quasiperiodic, or chaotic and to find a mapping of system attractors to performance measures. This can serve as a basis for the application of dynamic control methods, derived from Dynamical Systems Theory [2].

## 2 DYNAMICAL SYSTEMS CONCEPTS

A dynamical system is a system whose temporal evolution is caused by its internal parameters and by influences from its environment. The state of such a system varies over time. The Dynamical Systems Theory provides concepts and methods for modelling and analysing dynamical systems. Three of these concepts – the system state space, the trajectory, and the attractor – will be introduced in this chapter.

The current state of a dynamical system is determined by time-dependent system variables $x_1(t)$, $x_2(t)$, … $x_n(t)$. These variables span an n-dimensional state space where n is the number of the variables. Every system state $\boldsymbol{x}$ at time $t$ corresponds to a point in this state space determined by n coordinates $(x_1, x_2, … x_n)$. The temporal evolution of the system state leads to a series of points which create a curve – the so-called trajectory. Under certain conditions, all transient trajectories run towards a specific region of the state space. On can say, this region exerts an attractive force to the trajectories. It is therefore called attractor [3], [4].

The run of the trajectory through the state space and the topology of the system attractor visualise the dynamics of the system. This is shown by a simple example – the mathematical pendulum (Figure 1). Here, the ball of the pendulum is displaced by the angular displacement $\varphi$ and oscillates around its rest position with the angular velocity $\omega$. The state space of this pendulum is spanned by the variable $\varphi$ and its first derivative $\omega$ (Figure 2).

In a real experiment the oscillation of the pendulum is damped by friction. Thus the oscillation decreases until the pendulum reaches its rest position ($\varphi = 0$). Such a damped motion creates a spiral in the state space, starting at ($\varphi_0 = \varphi_{max}$, $\omega_0 = 0$) and running towards the point ($\varphi = 0$, $\omega = 0$) (Figure 2a). For all initial conditions ($\varphi_0$, $\omega_0$) this point exerts an attraction drawing all transient trajectories into it. It is therefore called 'point attractor'.
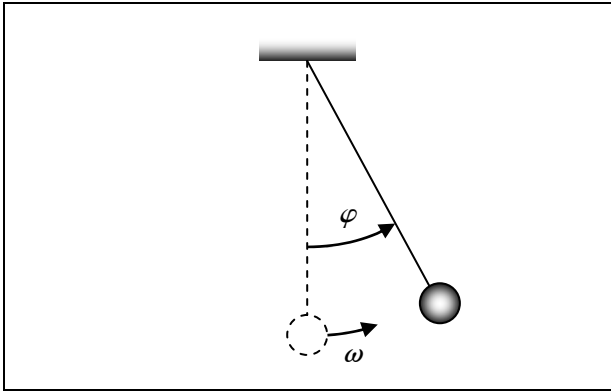
Figure 1: Dynamical system 'Pendulum'.

If the damping and driving forces on a pendulum are equal (e.g. in a pendulum clock), all transient trajectories will be drawn into a cycle, which has a specific diameter (Figure 2b). Such an attractor is called 'limit cycle'.

If the driving is stronger then the damping, the oscillation is amplified and the trajectory forms a spiral starting at ($\varphi_0 = 0$, $\omega_0 = 0$) and spinning towards $\varphi = 180°$ (Figure 2c). The pendulum then overturns and the motion changes from an oscillation to a rotation. Now the trajectory leaves the state space towards $\varphi \to \infty$. Thus, the driven pendulum has no attractor.

The attractor and its topology characterise the general dynamics of the considered system. The point attractor of the damped pendulum indicates a stable static system. The limit cycle attractor of the pendulum with balanced damping and driving forces indicates a stable periodic system. The divergent trajectory of the driven pendulum indicates an unstable system.

Further types of attractors can occur in state space of higher dimensionality. The trajectory of a quasiperiodic system runs on the surface of a 'torus'. If the quasiperiodic motion contains two incommensurate frequencies, the trajectory never reaches a point visited before, but completely fills the torus' surface. If the motion is chaotic, the trajectory of the system can form a 'strange attractor'. This is a fractal construct with a complicated but orderly geometrical form. For an introduction in chaotic systems and Chaos Theory, refer to [3] or [4].

The Dynamical Systems Theory provides several measures for the quantitative description of attractors.

Some of them are dimension measures. They describe the geometrical dimension of an attractor in the state space. In the pendulum example, the point attractor has the dimension D = 0 and the limit cycle has D = 1. The quasiperiodic attractor on the surface of a torus has the dimension D = 2. Chaotic attractors are characterised by a fractal dimension.

The system attractor is a fundamental object for studying a dynamical system. Its topology and the visited regions of the state space are important characteristics which describe the system's dynamics and provide possibilities for monitoring and controlling the dynamic behaviour of the system.

## 3 DYNAMICAL MODEL

The dynamical model of a manufacturing system introduced here is based on a deterministic queuing model by Hanson et al. [5] and uses the funnel metaphor of the work system model by Bechte [6]. Hanson et al. describe a reentrant manufacturing system by servers and queues, but use – contrary to stochastic queuing theory – deterministic parameters and variables to describe the system state. They introduce a state space and investigate the stability of their model using its trajectory. Bechte describes the work systems on the shop floor by funnels, where the funnel content represents the current work-in-process of the work system. Within the dynamical model introduced here, some relevant aspects of the model by Hanson et al. are integrated and enhanced for greater practical issues.

The manufacturing system considered here is a push-controlled flow line where a single part type is processed on a line of m work systems. Every work system consists of one machine $M_i$ and one input buffer $B_i$ (Figure 3). The machines $\mathbf{M} = (M_1 \ldots M_m)$ are modelled as funnels, where the liquid level represents the remaining processing time for the part in the machine. This variable describes the current state of the machines. It is denoted by $\boldsymbol{\tau} = (\tau_1 \ldots \tau_m)$, satisfying $T_i \geq \tau_i \geq 0$, where $T_i$ is the full processing time for a part at the machine $M_i$. The funnel outlets represent the processing capacities $\mathbf{C} = (C_1 \ldots C_m)$ which describe the actual throughput of the machines. They therefore define the actual processing time of each machine $M_i$ by $T_i = 1 / C_i$. The buffers $\mathbf{B} = (B_1 \ldots B_m)$ are modelled as queues. The current state of the buffers is described by the buffer levels $\boldsymbol{b} = (b_1 \ldots b_m)$, satisfying $b_{min,i} \leq b_i \leq b_{max,i}$, where $b_{min,i}$ and $b_{max,i}$, define the limits of buffer $B_i$.
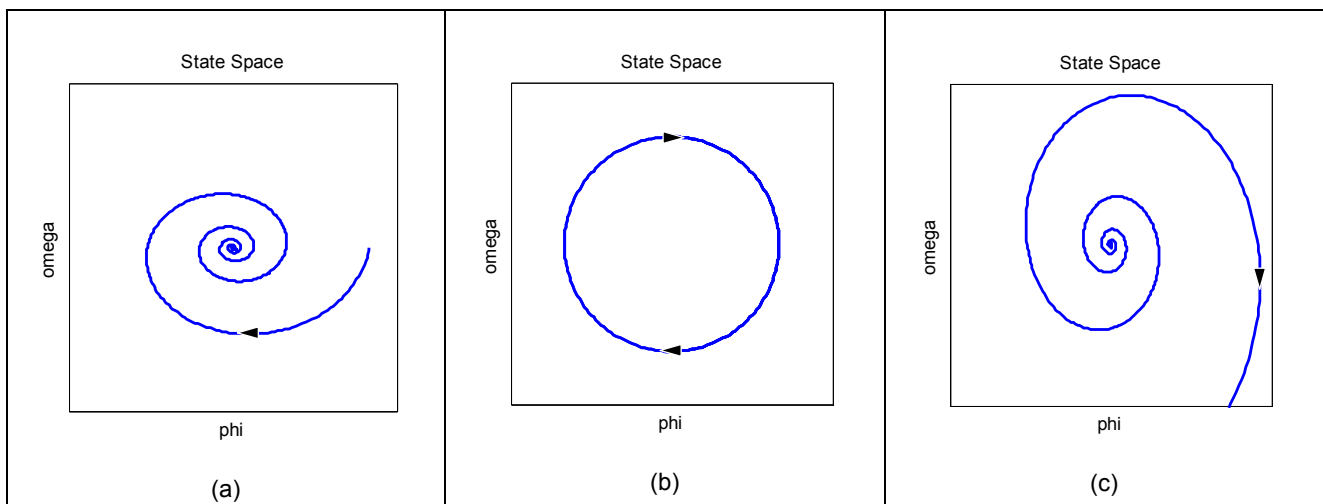


Figure 2: State spaces and trajectories of the (a) damped pendulum (point attractor), (b) ideal pendulum (limit cycle), (c) driven pendulum (divergent trajectory).
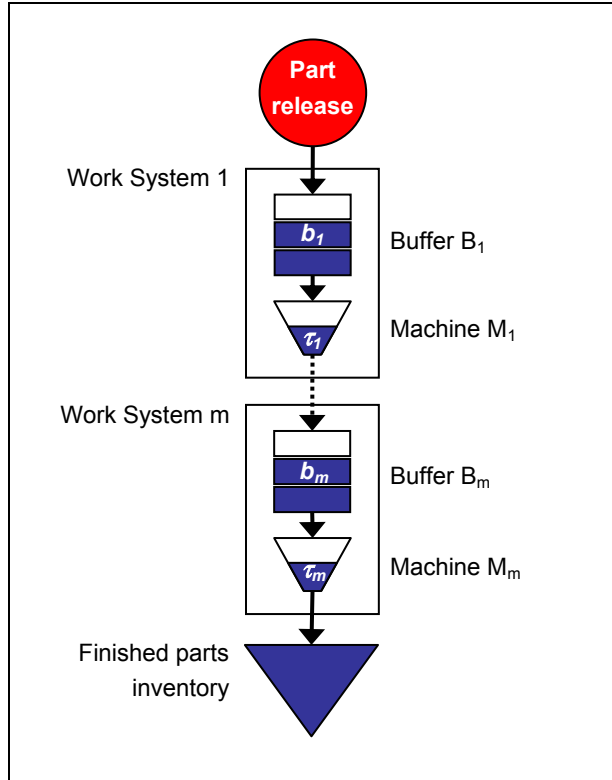
Figure 3: Push-controlled flow line.

Both variables $\tau$ and $b$ describe the state of the entire manufacturing system. They can be connected if the buffer levels are scaled to their required processing time and measured in hours work content [6]. Then the sum of both variables results in the work-in-process $wip = \tau + b$.

The part input into the system is dictated by the release rate $R = Q_R / T_R$, where $Q_R$ is the release quantity and $T_R$ is the release period. The dynamics of the input flow is described by the time it takes for the next part release to occur, denoted by $\tau_R$, satisfying $T_R \geq \tau_R \geq 0$.

According to [5], the dynamics of the part processing can be captured in an m-dimensional state space which is spanned by $\tau_1 \ldots \tau_m$. The size of this $\tau$ state space is given by the m processing times $T_1 \ldots T_m$. The trajectory $\tau$ is generally piecewise linear with a slope of -1. The visited regions within the state space thereby depend on the initial conditions and parameter settings. This is investigated in detail in section 4.2.

To capture the dynamics of the entire manufacturing system, a more comprehensive state space is necessary which is spanned by the m variables $wip_1 \ldots wip_m$. This $wip$ state space consists of $\tau$-subspaces, where every single subspace is associated with one of the possible values of $b$. The sizes of the $\tau$-subspaces are again given by the processing times $T_1 \ldots T_m$, the whole state space is limited by $b_{min}$ and $b_{max}$. The trajectory $wip$ is again piecewise linear with a slope of -1, whereas the linear pieces of the trajectory create a specific pattern in the $wip$ state space depending on the initial conditions and parameter settings. This is investigated in detail in section 4.3.

## 4 ANALYSIS OF DYNAMICS

### 4.1 Simple example of a manufacturing system

A simple manufacturing system consisting of two work systems is used to demonstrate the application of the introduced concepts state space, trajectory, and attractor and to analyse its dynamics.

The manufacturing system is described as follows:

Machines: $\mathbf{M} = (M_1, M_2)$,

Buffers: $\mathbf{B} = (B_1, B_2)$,

Buffer limits: $\mathbf{b}_{min} = (b_{min,1}, b_{min,2}) = \{0, 0\}$ parts,

$\mathbf{b}_{max} = (b_{max,1}, b_{max,2}) = \{9, 9\}$ parts,

Capacities: $\mathbf{C} = (C_1, C_2) = \{1, 1\}$ part/hour,

Processing times: $\mathbf{T} = (T_1, T_2) = \{1, 1\}$ hour/part,

Part release: $R = Q_R / T_R = 1$ part / 1 hour.

The variables are set to the following initial conditions:

$\boldsymbol{\tau}_0 = (\tau_{0,1}, \tau_{0,2}) = \{1 \cdot T_1, 1 \cdot T_2\} = \{1, 1\}$ hour,

$\mathbf{b}_0 = (b_{0,1}, b_{0,2}) = \{4, 3\}$ parts or hours work content,

$\tau_{R,0} = 1 \cdot T_R = 1$ hour.

### 4.2 Machining state space

The 2-dimensional machining state space is spanned by $\tau_1$ and $\tau_2$. Its size is given by $T_1$ and $T_2$ and amounts therewith $1\ h \times 1\ h$ (Figure 4). The run of the trajectory $\tau$ is determined by the initial conditions $\boldsymbol{\tau}_0$ and the ratio of the machine capacities $C_2/C_1$.

*Initial conditions*

At first, the influence of $\boldsymbol{\tau}_0$ is analysed using three different initial conditions:

(a): $\boldsymbol{\tau}_0 = \{1 \cdot T_1; 1 \cdot T_2\}$,

(b): $\boldsymbol{\tau}_0 = \{0.9 \cdot T_1; 0.7 \cdot T_2\}$,

(c): $\boldsymbol{\tau}_0 = \{0.9 \cdot T_1; 0.2 \cdot T_2\}$.

In case (a), both machines start their part processing at time $t = 0$ and finish after $T_1 = T_2 = 1$ h. The corresponding trajectory slopes from (1, 1) to (0, 0) and then jumps back to (1, 1) (Figure 4, solid blue line).

In case (b), the remaining processing times of $M_1$ and $M_2$ at $t = 0$ are 0.9 h and 0.7 h respectively. The corresponding trajectory slopes from (0.9, 0.7) to the lower limit of the state space at $\tau_1 = 0.2$ h and then jumps to the upper limit. From there, the trajectory slopes to the left limit of the state space at $\tau_2 = 0.8$ h and then jumps to the right limit. From here, the trajectory slopes to its starting point and repeats this way periodically (Figure 4, green dot-dashed line).
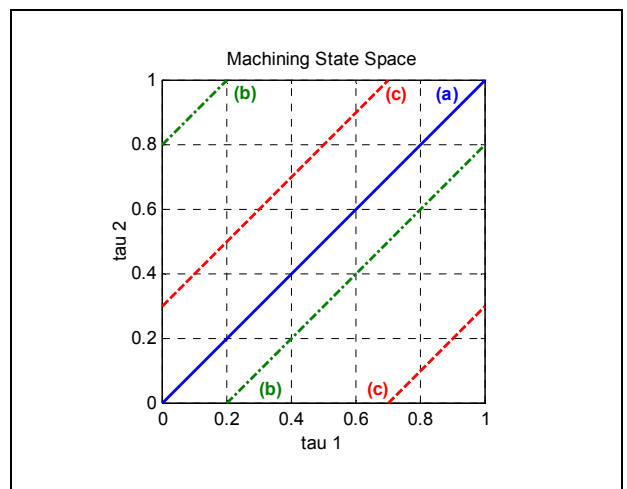


Figure 4: Influence of the initial conditions $\boldsymbol{\tau}_0$ on the run of the trajectory $\tau$: (a): $\varDelta\tau_0 = 0$, (b): $\varDelta\tau_0 = 0.2$ h, (c): $\varDelta\tau_0 = 0.7$ h.
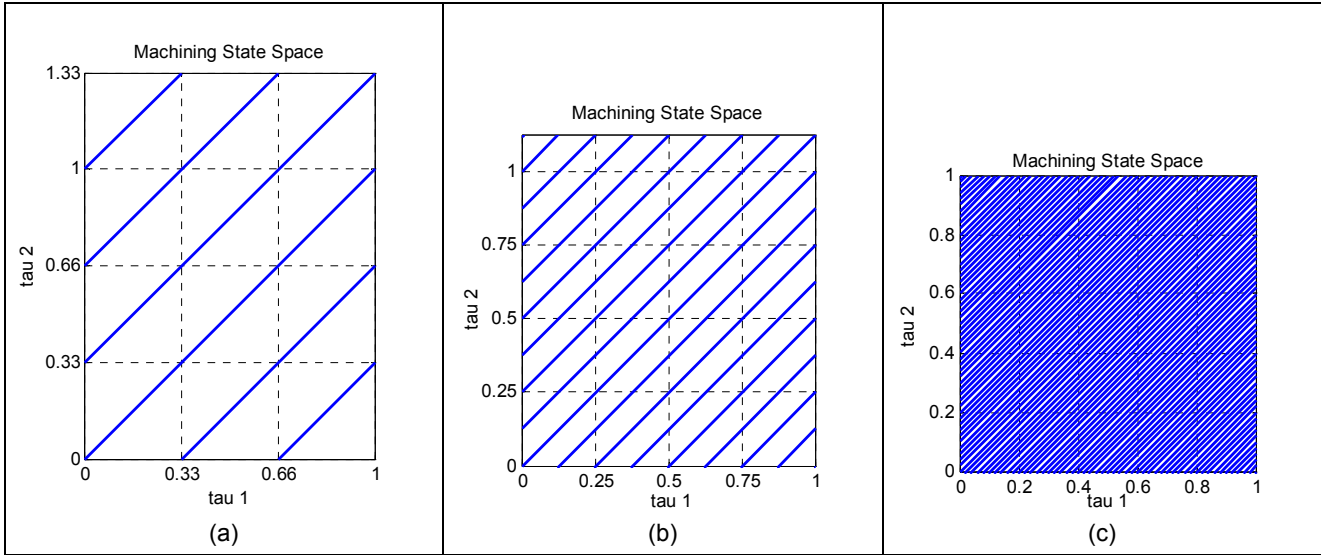
Figure 5: Influence of the ratio of the machine capacities $C_2/C_1$ on the run of the trajectory $\tau$:
(a): $C_2/C_1$ = 3/4, (b): $C_2/C_1$ = 8/9, (c) $C_2/C_1$ = 100/101.

The difference $\Delta\tau_0 = |\tau_{0,1} - \tau_{0,2}|$ = 0.2 h defines thereby the position of the saltus of the trajectory at $\Delta\tau_0$ and 1 – $\Delta\tau_0$.

In case (c) – analogous to case (b) – the difference of $\Delta\tau_0$ = 0.7 h causes jumps of the trajectory at $\tau_1$ = 0.7 h and $\tau_2$ = 0.3 h (Figure 4, red dashed line).

The different initial conditions cause only a shift of the trajectory $\tau$ within the state space. In all cases, the trajectory runs on an attractor which can be interpreted as a limit cycle with the dimension D = 1. So, the dynamics of the manufacturing system are always periodic with period 1.

The face of the state space changes if one of the machines becomes idle. Then, the attractor lies on the lower or the left limit respectively, but remains a limit cycle. But the dynamics change if both machines become idle. Then, the limit cycle collapses to a point attractor in (0, 0) with the dimension D = 0. This influence of idle times on the dynamics of a manufacturing system is investigated in detail by Hanson et al. [5].

*Parameter settings*

The influence of the machine capacities **C** is analysed using three different ratios $C_2/C_1$:

(a): $C_2$ = 3/4·$C_1$ = 0.75 parts/h,

(b): $C_2$ = 8/9·$C_1$ ≈ 0.89 parts/h,

(c): $C_2$ = 100/101·$C_1$ ≈ 0.99 parts/h.

Thereby, $C_1$ keeps its default value of 1 part/h (see section 4.1). Due to the relation $T_i$ = 1 / $C_i$, the processing time $T_2$ results in:

(a): $T_2$ ≈ 1.33 h/part,

(b): $T_2$ ≈ 1.125 h/part,

(c): $T_2$ ≈ 1.01 h/part.

So the height of the state space varies according to $T_2$ (Figure 5).

In case (a), the trajectory starts at (1, 1.33), slopes to the left limit of the state space and then jumps to the right one. From there, the trajectory slopes to the lower limit of the state space and then jumps to the upper one. This procedure recurs three times. At the end of this period 3 motion, the trajectory arrives at (0, 0) and jumps back from there to its starting point (1, 1.33). The ratio $C_2/C_1$ defines thereby the number and positions of the trajectory saltus (Figure 5a).

In case (b), the trajectory runs analogous to case (a) but with a higher saltus number. Here, the trajectory crosses the state space eight times until it arrives at its starting point (Figure 5b).

In case (a) and (b), the rational ratios $C_2/C_1$ cause different attractors with different periods. But in either cases, the trajectory runs on a limit cycle with the dimension D = 1.

Case (c) foretells the situation if the ratio $C_2/C_1$ becomes irrational. Then, the trajectory never reaches a point visited before, but completely fills the state space. The resulting attractor is an area with the dimension D = 2 (Figure 5c). This indicates possibly quasiperiodic dynamics at least in the introduced machining state space $\tau_1 \times \tau_2$.

In practice, the actual processing capacities vary due to machine failures, maintenance, etc. So the ratios of capacities vary too. This could temporarily lead to irrational capacity ratios – the dynamics of the manufacturing system could change qualitatively from periodicity to quasiperiodicity.

### 4.3 Wip state space

The 2-dimensional **wip** state space is spanned by $wip_1$ and $wip_2$. It is limited by $\mathbf{wip}_{min} = \mathbf{b}_{min}$ and $\mathbf{wip}_{max} = \mathbf{b}_{max}$ + **T** and amounts to 10 h × 10 h (Figure 8). The **wip** state space consists of 100 $\tau$-subspaces each associated with one of the possible values of **b** including **b** = (0, 0). The sizes of the $\tau$-subspaces are again dictated by the processing times $T_1$ and $T_2$ and thus amount to 1 h × 1 h. The run of the trajectory **wip** is determined by the initial conditions $\tau_0$, $\tau_{R,0}$, and $\mathbf{b}_0$, the parameters **C** and the control parameters R = $Q_R$ / $T_R$

*Initial conditions*

First, the influence of $\tau_0$ is analysed using four different initial conditions:

(a): $\tau_0$ = {1·$T_1$; 1·$T_2$},

(b): $\tau_0$ = {1·$T_1$, 0.7·$T_2$},

(c): $\tau_0$ = {0.9·$T_1$, 1·$T_2$},

(d): $\tau_0$ = {0.9·$T_1$, 0.7·$T_2$}.

Figure 6: Influence of the initial conditions $\tau_0$ on the run of the trajectory **wip**: (a): $\tau_0 = \{1 \cdot T_1; 1 \cdot T_2\}$, (b): $\tau_0 = \{1 \cdot T_1, 0.7 \cdot T_2\}$, (c): $\tau_0 = \{0.9 \cdot T_1, 1 \cdot T_2\}$, (d): $\tau_0 = \{0.9 \cdot T_1, 0.7 \cdot T_2\}$.
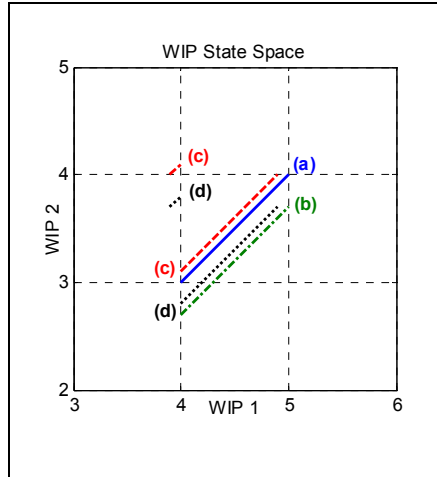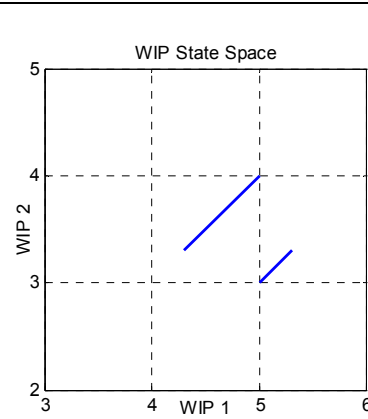
Figure 7: Influence of the initial condition $\tau_{R,0}$ on the run of the trajectory **wip**: $\tau_{R,0} = 0.7 \cdot T_R$.
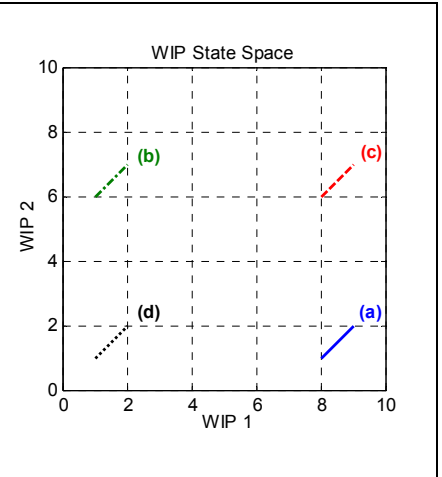
Figure 8: Influence of the initial conditions $b_0$ on the run of the trajectory **wip**: $b_0 = \{8, 1\}$ parts, (b): $b_0 = \{1, 6\}$ parts, (c): $b_0 = \{8, 6\}$ parts, (d): $b_0 = \{1, 1\}$ parts.

Due to the initial conditions $b_0 = (b_{0,1}, b_{0,2}) = \{4, 3\}$ h, the trajectory moves within and around the $\tau$-subspace dedicated to these buffer levels. Here, the left lower corner indicates the associated buffer levels and the subspace above indicates the parts in the machines (Figure 6).

In case (a), both machines start their part processing at time $t = 0$ and finish after $T_1 = T_2 = 1$ h. The corresponding trajectory slopes from (5, 4) to (4, 3) and then jumps back to (5, 4) (Figure 6, solid blue line).

In case (b), the remaining processing times of $M_1$ and $M_2$ at $t = 0$ are 1 h and 0.7 h respectively. The corresponding trajectory slopes from (5, 3.7) to (4, 2.7). The difference $\Delta\tau_0 = \tau_{0,1} - \tau_{0,2} = 0.3$ h causes a vertical shift of the trajectory downward (Figure 6, green dot-dashed line).

In case (c), the remaining processing times of $M_1$ and $M_2$ at $t = 0$ are 0.9 h and 1 h respectively. The corresponding trajectory slopes from (4.9, 4) to the left limit of the $\tau$-subspace and then jumps vertically upward. The difference $\Delta\tau_0 = \tau_{0,1} - \tau_{0,2} = -0.1$ h causes a horizontal shift of the trajectory to the left and a saltus at the subspace limit (Figure 6, red dashed line). This jump for $\Delta wip_2 = +1$ h is caused by the transition of a finished part from machine $M_1$ to the buffer $B_2$. These saltus at the vertical subspace limit $wip_i = n \cdot T_i$ ($n \in \mathbb{N}$) are a general characteristic of the manufacturing system at hand.

Finally, case (d) is a combination of cases (b) and (c) (Figure 6, black dotted line).

The considered differences in the processing starts ($\Delta\tau_0 \neq 0$) represent the asynchronous operation of the machines in a flow line, where the machines are decoupled by buffers. The necessity of buffers will be obvious in the **wip** state space. The trajectory crosses at least two different $\tau$-subspaces which is caused by a change in **b** (Figure 6b-d). Contrary to this, a synchronous flow line ($\Delta\tau_0 = 0$) does not need buffers, because the trajectory never leaves the initial $\tau$-subspace (Figure 6a). So **b** could be set on (0, 0) and will never change.

At next, the influence of the part release is analysed using the initial waiting time $\tau_{R,0}$. The default setting $\tau_{R,0} = 1 \cdot T_R$ means that a part release occurs at $t = \{0, 1 \cdot T_R, 2 \cdot T_R, \ldots\}$. Here, the run of the trajectory **wip** corresponds to Figure 6a. A change of the initial waiting time $\tau_{R,0} =$

$0.7 \cdot T_R$ causes a shift of the release sequence to $t = \{0.7 \cdot T_R, 1.7 \cdot T_R, \ldots\}$. The resulting trajectory is shown in Figure 7. The saltus at $\tau_1 = \tau_{1,0} - \tau_{R,0} = 0.3$ h to $\Delta wip_1 = +1$ h is caused by the release of a new part and the inflow to buffer $B_1$ at $t = 0.7$ h.

Finally, the influence of $b_0$ on the location of the trajectory **wip** is analysed using four different initial conditions:

(a): $b_0 = \{8, 1\}$ parts,

(b): $b_0 = \{1, 6\}$ parts,

(c): $b_0 = \{8, 6\}$ parts,

(d): $b_0 = \{1, 1\}$ parts.

The trajectories generally start at $b_0 + \tau_0$ and slope to $b_0$ (Figure 8).

In sum, all changes in initial conditions $\tau_0$, $\tau_{R,0}$ and $b_0$ influence only the location of the trajectory in the **wip** state space. The general periodic dynamics do not change.

*Parameter settings*

The influence of the work load on the appearance of the **wip** state space is analysed using different settings of **C**. Thereby, the part input into the system keeps constant at $R = 1$ part/h. Due to the relation $T_i = 1 / C_i$, the processing times and therewith the sizes of the $\tau$-subspaces and the overall **wip** state space vary from case to case. The initial buffer levels are set to $b_0 = \{5, 5\}$ parts to shift the starting point of the trajectory to the middle of the state space.

At first, an underloaded system is investigated using the following processing capacities:

(a): $C_1 = 10/8$ parts/h, $C_2 = \{11/8, 10/8, 9/8\}$ parts/h,

(b): $C_1 = \{4/4, 5/4\}$ parts/h, $C_2 = \{5/4, 4/4\}$ parts/h.

The resulting trajectories are shown in the figures 9a and 9b. They form a pattern depending on the ratio $C_2/C_1$ (cp. Figure 5) and on the ratio $R/C_i$. But in all cases, the trajectory first runs towards the left or lower limit of the state space, and finally arrives at point (0, 0). The behaviour of the underloaded system can be compared with the damped pendulum, where the dynamics change from periodicity (limit cycle) to a static state (point attractor).
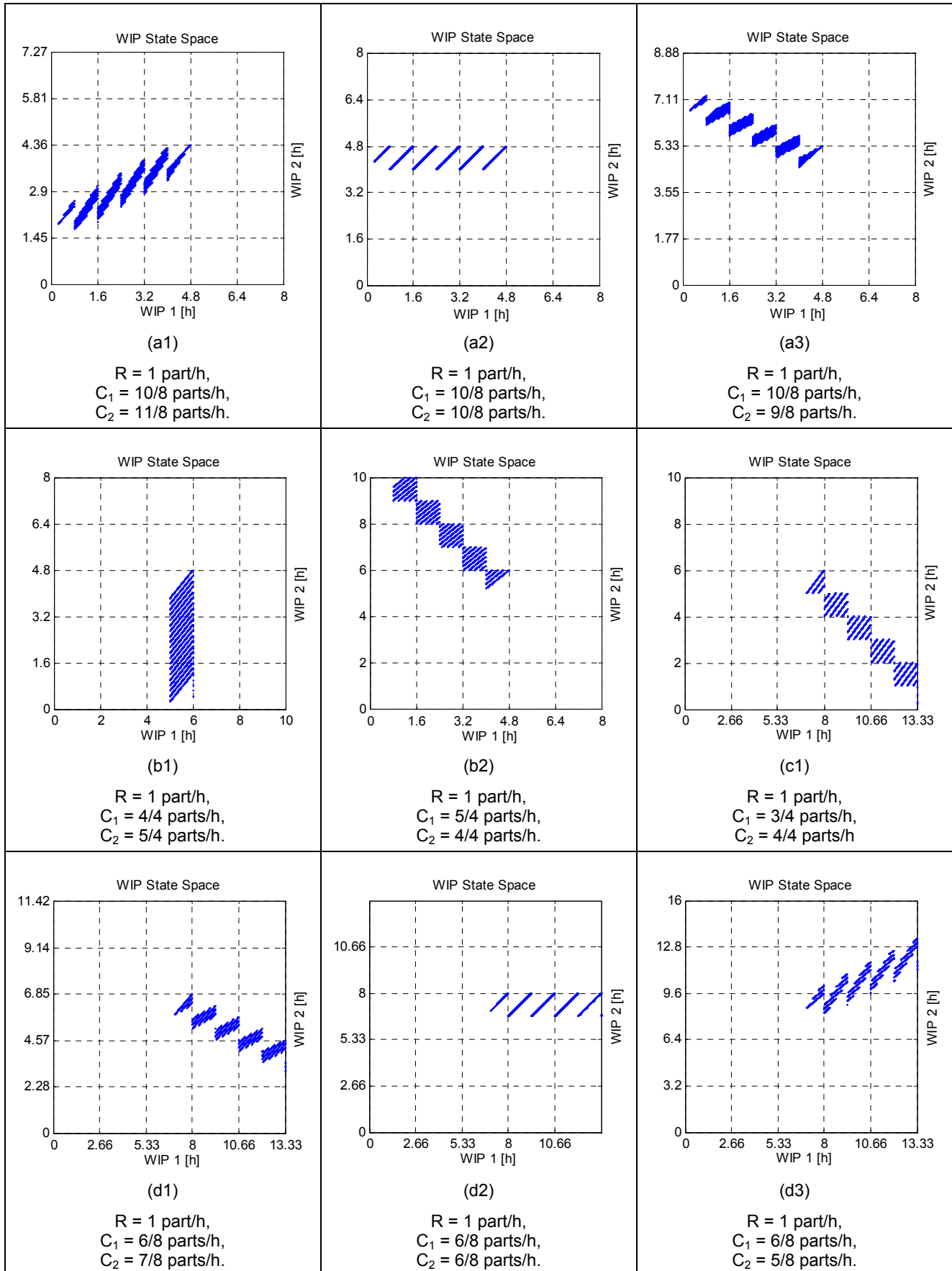
(a1)

R = 1 part/h,
$C_1$ = 10/8 parts/h,
$C_2$ = 11/8 parts/h.

(a2)

R = 1 part/h,
$C_1$ = 10/8 parts/h,
$C_2$ = 10/8 parts/h.

(a3)

R = 1 part/h,
$C_1$ = 10/8 parts/h,
$C_2$ = 9/8 parts/h.

(b1)

R = 1 part/h,
$C_1$ = 4/4 parts/h,
$C_2$ = 5/4 parts/h.

(b2)

R = 1 part/h,
$C_1$ = 5/4 parts/h,
$C_2$ = 4/4 parts/h.

(c1)

R = 1 part/h,
$C_1$ = 3/4 parts/h,
$C_2$ = 4/4 parts/h

(d1)

R = 1 part/h,
$C_1$ = 6/8 parts/h,
$C_2$ = 7/8 parts/h.

(d2)

R = 1 part/h,
$C_1$ = 6/8 parts/h,
$C_2$ = 6/8 parts/h.

(d3)

R = 1 part/h,
$C_1$ = 6/8 parts/h,
$C_2$ = 5/8 parts/h.

Figure 9: Influence of work load $R/C_i$ on the run of the trajectory **wip**.
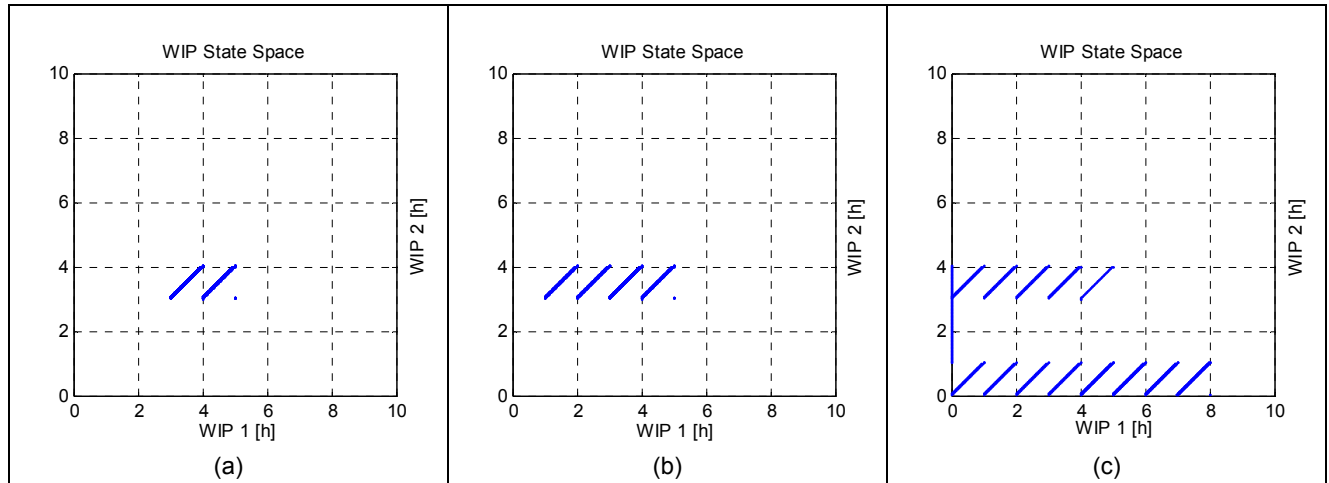
Figure 10: Influence of release policy on the run of the trajectory **wip**: (a): $Q_R$ = 2 parts, $T_R$ = 2 h, (b): $Q_R$ = 4 parts, $T_R$ = 4 h, (c): $Q_R$ = 8 parts, $T_R$ = 8 h.

The balanced system corresponds to the default setting and is analysed in all the previous cases. The resulting trajectory is a simple line in a single $\tau$-subspace (cp. e.g. Figure 6a). These dynamics can be compared with the periodic dynamics of the ideal pendulum.

Finally, an overloaded system is investigated using the following processing capacities:

(c): $C_1$ = {4/4, 3/4} parts/h, $C_2$ = {3/4, 4/4} parts/h,

(d): $C_1$ = 6/8 parts/h, $C_2$ = {7/8, 6/8, 5/8} parts/h.

The resulting trajectories are shown in the figures 9c and 9d. They form similar patterns as in cases (a) and (b) but the direction of the run is contrary to these. Every trajectory runs towards the right or upper limit of the state space and leaves it towards **wip** $\rightarrow \infty$. The behaviour of the overloaded system can be compared with the driven pendulum where the dynamics are unstable and form a divergent trajectory.

By using the metaphor of the dynamical system 'pendulum' more stringently, the machine capacities can be considered as damping forces in the manufacturing system whereas the part input into the system can be interpreted as its drive. Thus, the last analysis is dedicated to the influence of the part release on the appearance of the **wip** state space.

The part input into the system is given by default as R = $Q_R$ / $T_R$ = 1 part / 1 hour. This means, one part will be released every hour. This input rate fits exactly the machine capacities – the system is balanced. This balance persists as long as the release quantity $Q_R$ and the release period $T_R$ have the same values.

The influence of quantity and period of part release in such a balanced system is analysed using three different release polices:

(a): $Q_R$ = 2 parts, $T_R$ = 2 h,

(b): $Q_R$ = 4 parts, $T_R$ = 4 h,

(c): $Q_R$ = 8 parts, $T_R$ = 8 h.

In case (a), the release of 2 parts every 2 hours causes a period 2 motion, indicated by two lines in the state space (Figure 10a).

According to this, case (b) leads to an oscillation with period 4 (Figure 10b).

Case (c) seems to be different (Figure 10c). The trajectory runs towards the left limit of the state space and then falls down along the axis $wip_2$. But before the trajectory reaches the point (0, 0), 8 parts are released and the trajectory jumps to the right and starts its regular run with period 8. So the first pieces of the trajectory between $wip_2$ = 3 … 4 represent the transient part, whereas the following pieces between $wip_2$ = 0 … 1 form the attractor of the system according to cases (a) and (b).

In sum, the different release policies cause different attractors. Thereby, the release period is responsible for the period of the attractor and the release quantity determines the visited $\tau$-subspaces within the **wip** state space. However, the general periodic dynamics do not change under the investigated push strategy.

## 5 DYNAMICS VERSUS PERFORMANCE

Manufacturing systems are usually appraised not by their dynamics but by their performance. Common performance measures are the total work-in-process and its costs, the throughput volume and the throughput time. The total work-in-process and the throughput time are related to each other by Little's Law: Work-in-process = release rate of parts $\times$ throughput time. Accordingly, low work-in-process minimises not only costs but also shortens the throughput time of a part. Short and predictable throughput times ensure meeting due-dates, allow for better planning of releases into the system and lead to better coordination of further operations such as assembly.

Little's Law can be used to control a manufacturing system, whereas the release rate serves as control parameter and the work-in-process allows for monitoring the system to observe the control results.

Within the dynamical model, the work-in-process is a time-dependent variable which captures the dynamic behaviour of a manufacturing system. So, dynamics and performance are directly linked to each other by the work-in-process **wip**. The introduced **wip** state space serves as a monitor for observing the system dynamics as well as the system performance. Here, the run of the trajectory **wip**, the visited $\tau$-subspaces and the location of the attractor represent a certain performance. Unfortunately, it is difficult to find a quantitative mapping from the system attractors to the system performance. But it is possible to define $\tau$-subspaces representing good or poor performance. Furthermore, it is possible to assign certain $\tau$-subspaces to specific control actions. This is shown in figure 11.

The centre of the white region represents the work-in-process necessary to decouple the machines from each other to manage uncertainties such as machine failures, emergency maintenances, etc.
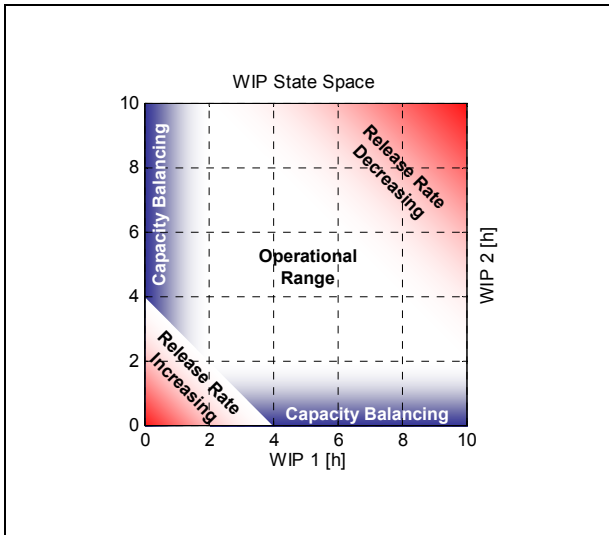
Figure 11: WIP control in state space.

The right upper corner represents high work-in-process leading to long throughput times and a poor performance. As a quick intervention, the release rate should be decreased. A long-term strategy could be an increase in system capacity. Contrary to this, the left lower corner represents low work-in-process leading to idle times and poor capacity utilisation. As a quick intervention, the release rate should be increased. A long-term strategy could be a decrease in system capacity. A trajectory close to the left or lower limit of the state space indicates an uneven load of the different work systems. Here, capacity balancing is necessary to avoid idle times and bottlenecks.

Figure 11 depicts a qualitative map of attractor regions representing a certain performance of the manufacturing system. This map is useful to find parameter settings leading to low work-in-process and short throughput times. The resulting attractors then represent good system performance.

## 6 SUMMARY AND OUTLOOK

This paper presented a dynamical systems perspective on manufacturing systems. A dynamical model of a manufacturing system was introduced using concepts of Dynamical Systems Theory. The dynamics of this model are described by the variables 'buffer level' and 'remaining processing time'. Both variables represent the work-in-process of a work system which is also considered a time-dependent variable *wip*. This variable spans an m-dimensional state space where m is the number of work systems. The temporal evolution of the system is described by a trajectory, which is generally piecewise linear with a slope of -1.

The work systems are compared to a pendulum oscillator, which are driven by the part input and damped by their processing capacity. Thereby, the oscillation

period of the pendulum corresponds to the processing time for one part at a machine.

Depending on the ratio between part input and capacity, the underloaded manufacturing system corresponds to the damped pendulum, whereas the overloaded system behaves like a driven pendulum. The balanced manufacturing system corresponds to an ideal pendulum, where driving and damping forces are equal.

The dynamics of the manufacturing system seems to be generally periodic. However, if the ratios between the capacities of different machines take irrational values, the resulting attractor indicates possibly quasiperiodic dynamics.

The analysis results shown in this paper are just a first step toward a more extensive investigation on the dynamics of manufacturing systems. Further work could carry on the qualitative analysis to investigate, e.g. the influence of different control strategies, different queuing policies or more complex structures on the dynamics of the manufacturing system. Furthermore, a quantitative description of the dynamics will be necessary to benefit from Dynamical Systems Theory in order to understand and control the dynamics of a manufacturing system.

## 7 ACKNOWLEDGMENTS

## 8 REFERENCES

[1] Beaumariage, T., Kempf, K., 1994, The Nature and Origin of Chaos in Manufacturing Systems, IEEE/SEMI Advanced Semiconductor Manufacturing Conference, pp. 169-174.

[2] Scholz-Reiter, B., Freitag, M., Schmieder, A., 2002, Modelling and Control of Production Systems based on Nonlinear Dynamics Theory, CIRP Annals 51/1, pp. 375-378.

[3] Ott, E., 1993, Chaos in Dynamical Systems, Cambridge University Press.

[4] Strogatz, S. H., 1994, Nonlinear Dynamics and Chaos, Addison-Wesley.

[5] Hanson, D., Armbruster, D., Taylor, T., 1998, On the Stability of Reentrant Manufacturing Systems, Proceedings of the 31st MTNS, pp. 937-940.

[6] Bechte, W., 1984, Steuerung der Durchlaufzeit durch belastungsorientierte Auftragsfreigabe bei Werkstattfertigung, Fortschritt-Berichte VDI, Reihe 2, Nr. 70, Düsseldorf.