Optimal trade execution in order books with stochastic liquidity∗

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Abstract

In financial markets, liquidity changes randomly over time. We consider such random variations of the depth of the order book and evaluate their influence on optimal trade execution strategies. If the stochastic structure of liquidity changes satisfies certain conditions, then the unique optimal trading strategy exhibits a conventional structure with a single wait region and a single buy region and profitable round trip strategies do not exist. In other cases, optimal strategies can feature multiple wait regions and optimal trade sizes that can be decreasing in the size of the position to be liquidated. Furthermore, round trip strategies can be profitable depending on bid-ask spread assumptions. We illustrate our findings with several examples including the CIR model for the evolution of liquidity.


1 Introduction

Liquidity is not constant throughout the day, but instead varies over time. Traders active in a market are typically expected to continuously observe these changes in liquidity and adjust their trades accordingly. Some part of the liquidity changes is driven by deterministic changes in expected liquidity levels, e.g., daily and weekly patterns as well as expected changes around important points in time such as news releases. These expected changes however do not explain liquidity variation fully. An unpredicted component of liquidity changes remains which can dominate the deterministic component.

We extend existing limit order book models and introduce a stochastic depth of the order book. In this market, we consider an investor who wants to purchase a large asset position. If the order book dynamics are driven by a general diffusion satisfying certain conditions, then we prove existence and uniqueness of the optimal trade execution strategy. This trading strategy exhibits a wait region / buy region structure with a single wait region and a single buy region. If the investor finds herself in the wait region at a given point in time, then she does not place any orders at this point; if she is in the buy region, then the investor buys just enough to bring her position from within the buy region to the boundary of the wait region.

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If limit order book depth is not driven by a diffusion satisfying said conditions, then the classical wait region / buy region structure with one region each does not need to hold. While optimal strategies may still be described in terms of wait and buy regions, there can be more than one of these regions. We provide several examples with such non-standard optimal trading strategies. Intuitively expected features do not need to hold any more. For example, the trade size at a given point in time can vary non-monotonically with the size of the remaining position: if a large or small position remains, then no order is placed, however a purchase order is placed if the remaining position is of medium size. To the best of our knowledge, a nonintuitive structure of such type in solutions of Markovian control problems was never observed in the literature. In particular, this concerns also our previous paper Fruth, Schöneborn, and Urusov (2014) on the subject, where the results imply that such a phenomenon cannot appear whenever the order book depth is a deterministic function of time.

In our model, the condition ensuring wait region / buy region structure also guarantees that round trip trading strategies cannot be profitable. If the condition is violated, then round trip strategies can generate profits if the bid-ask spread is assumed to be zero; if a dynamic spread is assumed, then profits from round trip strategies remain unavailable.

The majority of the optimal trade execution literature considers one of two different market models. First, several models assume an instantaneous temporary price impact, e.g., Almgren and Chriss (2001) and Almgren (2003). In these models, the temporary price impact at time $t$ is independent of all orders executed at time prior to $t$ and does not influence any order at a time after $t$, which greatly simplifies the analysis. Cheridito and Sepin (2014) and Almgren (2012) have studied stochastic temporary price impact in this setting and provide numerical methods for calculation of the optimal strategy and value function, while Ankirchner, Jeanblanc, and Kruse (2014) and Ankirchner and Kruse (2015) add an additional quadratic risk term and provide the solution in terms of a BSDE with a singular terminal condition. Also see Dolinsky and Soner (2013), Bank, Dolinsky, and Göökay (2016) for the discussion of super-replication as well as Bank, Soner, and Voß (2017) for the discussion of hedging in such a setting with stochastic temporary price impact.

In a second group of models, inspired by a limit order book interpretation, resilience is finite and depth and resilience are separately modelled. Our model falls into this second group. Due to the finite resilience of the order book, the execution price at time $t$ is influenced by orders filled at times prior to $t$, and the execution at time $t$ in turn influences the execution price of subsequent orders. In our present paper, due to the fact that the order book depth is stochastic, the optimal strategies are carried on a set of Lebesgue measure zero (we obtain a stop-and-go pattern depending on whether we are in the interior or on the boundary of the wait region). This is in contrast to the optimal strategies in the first group of papers considering (stochastic) temporary impact, which are absolutely continuous with respect to the Lebesgue measure. Most of the existing literature on order book driven optimal liquidation assumes the liquidity parameters to be constant over time, see, e.g., Bouchaud, Gefen, Patters, and Wyart (2004), Obizhaeva and Wang (2013), Alfonsi, Fruth, and Schied (2010) and Predoiu, Shaikhet, and Shreve (2011). Alfonsi and Acevedo (2014), Bank and Fruth (2014) and Fruth, Schöneborn, and Urusov (2014) allow for deterministic changes in liquidity and are therefore closely related to our paper. This paper is qualitatively different from the aforementioned papers, and the main differences are as follows. Due to the stochasticity in the depth of the order book, the optimal execution strategies in the framework of this paper are no longer deterministic (the latter was the case in the aforementioned group of papers, see also Section 2). More surprisingly, the counterexamples to the wait region / buy region structure mentioned above appear in the framework of our present paper only.

Another paper that considers stochastically varying order book depth is Chen, Kou, and Wang (2015). They provide a numerical method for calculation of the optimal strategy and value function in discrete time with the depth of the limit order book driven by a discrete Markov chain. In contrast, we focus on analytical results in a continuous time setting with limit order book depth following a diffusive process.
Becherer, Bilarev, and Frentrup (2017) analyse yet another version of stochastic liquidity in a framework that resembles the framework in the aforementioned second group of models. They have a certain impact process \( Y \), which is analogous to our deviation process \( D \) (see (2)) with a constant resilience \( \rho_s \equiv \beta \in (0, \infty) \), a constant liquidity process \( (K_s) \) normalised to one and additional Brownian noise. Stochasticity of liquidity is introduced in their model due to the mentioned Brownian noise as well as due to a multiplicative price impact built via a nonlinear transform applied to the process \( Y \) (in contrast, we have an additive price impact in our modelling approach). Becherer, Bilarev, and Frentrup (2017) find an explicit solution of the singular control problem, which corresponds to optimal execution with infinite time horizon. Even though both in our present paper and in Becherer, Bilarev, and Frentrup (2017) the optimal strategies are described via singular control problems, they are qualitatively different in that they even require different state spaces, which is due to the modelling differences.

A modelling framework that comprises instantaneous temporary price impact as in Almgren (2012), transient (or persistent) price impact like the one generated by a limit order book, and a certain risk term that penalises slow liquidation is considered in Graewe and Horst (2016). While the part of the dynamics in Graewe and Horst (2016) that is linked to a limit order book corresponds to a constant (deterministic) order book depth, they have stochastic resilience. In this non-Markovian situation, they prove that the value function is characterised as a unique solution to a three-dimensional BSDE system. The differences with our approach are as follows. Firstly, as Graewe and Horst (2016) require a nondegenerate instantaneous price impact, their optimal strategies are, as in Almgren (2012), absolutely continuous with respect to the Lebesgue measure. Secondly, when we let the instantaneous price impact go to zero in their model, we can obtain the optimal strategy of Obizhaeva and Wang (2013) in the limit, which corresponds to a constant order book depth, but we cannot get in this way the strategies corresponding to genuinely stochastic order book depth.

Starting with Huberman and Stanzl (2004), profitable round trip strategies have been studied in a variety of market models by Gatheral (2010), Alfonsi and Schied (2010), Alfonsi, Schied, and Slynko (2012) and Klöck, Schied, and Sun (2014) among others. To the best of our knowledge, all existing literature on this topic assumes deterministic liquidity, while in our paper we consider it in our stochastic liquidity model.

The remainder of this paper is structured as follows. We first compare this paper with our previous paper Fruth, Schöneborn, and Urusov (2014) in Section 2. In Section 3, we introduce a limit order book model with stochastic depth and derive basic structural features in Section 4. We prove existence and uniqueness of optimal strategies as well as the wait region / buy region structure in Section 5 as long as the stochastic dynamics of the limit order book depth obeys certain conditions. We apply these results to several examples of diffusive processes in Section 6. If the conditions of Section 5 are violated, then the optimal strategy does not need to be of wait region / buy region structure any more as we demonstrate in several examples in Section 7. In Section 8, we extend our model to two-sided limit order books and investigate the returns of round trip trading strategies. We conclude in Section 9 and provide proofs in Appendices A, B, C, D and E.

2 Connection to our previous work

This paper considers a liquidity model that extends the model examined in our previous paper Fruth, Schöneborn, and Urusov (2014). In this section, we discuss the connection between these two papers.

Model: In our previous paper Fruth, Schöneborn, and Urusov (2014) we have analyzed an order book model where the order book depth is assumed to be a deterministic function of time. The current paper extends the previous one by allowing the order book depth to be stochastic. This corresponds

\[ 1^\text{Many (but not all) of the results of both this and our previous paper are also contained in the PhD thesis Fruth (2011).} \]
to an additional dimension in our optimization problem, which we introduce in all notations. The remainder of the model is identical in both papers.

We now compare the main results in the present paper to the main results in Fruth, Schöneborn, and Urusov (2014) in the two main topics of this work:

**Topic 1: Existence, uniqueness and structure of optimal strategies.** In both papers we prove existence and uniqueness of an optimal trading strategy that has an intuitive wait region / buy region structure. In our previous paper we could show this under relatively general assumptions on the deterministic liquidity process. In the stochastic liquidity framework of the current paper, we need more restrictive assumptions to be able to derive the wait region / buy region structure. The techniques that we need are completely different to the ones in the deterministic case. Subsequently we construct examples of stochastic liquidity processes for which the intuitive wait region / buy region structure does not hold. For examples where it does hold, we numerically derive optimal strategies which are of course no longer deterministic (as in the case of deterministic liquidity).

**Topic 2: Profitable round trip trading strategies.** In our previous paper we introduced the dynamic and zero spread models for the two-sided order book and study existence of profitable round trip strategies in these models. The existence of profitable round trip strategies is sometimes also referred to as price manipulation. For a deterministic liquidity process, there is no price manipulation in the dynamic spread model, while in the zero spread model profitable round trips can occur depending on the model parameters. In the present paper we study the same questions in the framework of stochastic liquidity. Here, again, there is no price manipulation in the dynamic spread model, and the line of argument from Fruth, Schöneborn, and Urusov (2014) applies directly. For the zero spread model however we now need a different approach and discover a new connection between the absence of profitable round trip strategies and the existence of wait region / buy region structure.

While this paper is self-contained, we refer the interested reader to our previous paper where we provide more detail on financial motivation.

## 3 Model description

A limit order book model with time dependent depth was introduced in Fruth, Schöneborn, and Urusov (2014). In this previous paper we explain the model in depth and provide an economic motivation. In the following, we recapitulate the central components and notation and extend the model from deterministic order book depth to stochastic order book depth.

The model is built on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0, T]}, \mathbb{P})\). As usual in dynamic programming we consider a general initial time \(t \in [0, T]\) below. For the evolution of the trader’s asset position over time interval \([t, T]\), we consider the set of *admissible* continuous-time increasing strategies

\[
A^{cts}_t := \{ \Theta : \Omega \times [t, T+] \rightarrow [0, \infty) | (\mathcal{F}_s) - \text{adapted, increasing, bounded, càglàd with } \Theta_t = 0 \} \]

and denote \(\xi := \Delta \Theta_s := \Theta_{s+} - \Theta_s\). In particular, absolutely continuous trading as well as impulse trades are allowed. A strategy from \(A^{cts}_t\) consists of a left-continuous process \((\Theta_s)_{s \in [t, T]}\) and an additional random variable \(\Theta_T\), with \(\Delta \Theta_T = \Theta_{T+} - \Theta_T \geq 0\) being the last trade of the strategy. Let us emphasize that admissible strategies are bounded by definition, that is, for \(\Theta \in A^{cts}_t\), we have \(\Theta_{T+} \leq \text{const} < \infty\) a.s. (the constant depends on a strategy). Denote by

\[
A^{cts}_t(x) := \{ \Theta \in A^{cts}_t | \Theta_{T+} = x \text{ a.s.} \} \quad (1)
\]

the admissible strategies that build up a position of \(x \in [0, \infty)\) shares until time \(T\) almost surely. For the majority of this paper, we consider only one side of the limit order book (namely, the buy side).
and hence only include increasing strategies in $\mathcal{A}^\text{cts}_t$. As we will see in Section [8], selling cannot reduce overall purchase costs if the bid-ask spread is influenced by the trader.

In addition to continuous time, we will also consider trading in discrete time, i.e., at times

$$0 = t_0 < t_1 < \ldots < t_N = T.$$

In this case, we constrain our admissible strategy set to

$$\mathcal{A}^\text{dis}_t := \{ \Theta \in \mathcal{A}^\text{cts}_t | \Theta_s = 0 \text{ on } [t, t_n(t)] \text{ and } \Theta_s = \Theta_{t_n^+} \text{ a.s. on } (t_n, t_{n+1}) \text{ for } n = n(t), \ldots, N - 1 \} \subset \mathcal{A}^\text{cts}_t$$

with $n(t) := \inf\{n = 0, \ldots, N | t_n \geq t\}$ and define

$$\mathcal{A}^\text{dis}_t(x) := \{ \Theta \in \mathcal{A}^\text{dis}_t | \Theta_{T^+} = x \text{ a.s.} \}$$

as the discrete analogue to $\mathcal{A}^\text{cts}_t(x)$.

Let $D$ be an “ask dislocation” process, i.e., the deviation of the current ask price from its steady state level, $K$ the illiquidity process, and $\rho$ the (time-varying) resilience speed.

**Standing Assumption.**

(i) $K$ is a (possibly time-inhomogeneous) $(\mathcal{F}_s)$-Markov process with state space $(0, \infty)$ and finite first moments.

(ii) $\rho: [0, T] \to (0, \infty)$ is a strictly positive Lebesgue-integrable deterministic function.

We consider a block-shaped order book, where the height of the block (≡ order book depth) at each time $s$ is $q_s := 1/K_s$. If we, for instance, buy $y$ shares at time $s_0$, the deviation increases by $\frac{y}{q_{s_0}} = K_{s_0}y$ and therefore $D_{s_0^+} = D_{s_0} + K_{s_0}y$. If we do not trade for some time afterwards, the deviation decays (moves towards zero), which is a feature of real-life limit order books called *resilience*. We assume exponential resilience in our model, that is, if we do not trade on some time interval $(s_0, s_1)$, we arrive at the deviation $D_{s_1} = D_{s_0} + K_{s_0} \exp\left\{-\int_{s_0}^{s_1} \rho_r \, dr\right\} y$. The following formulas (2)–(4) show the interplay of the effects from buying shares ($D$ increases) and from the resilience ($D$ decreases) in our model for a general strategy $\Theta$. The deviation $D_s$ results from past trades on $[t, s]$ in the following way

$$dD_s = -\rho_s D_s ds + K_s d\Theta_s, \quad D_t = \delta.$$  

That is, for $s \in [t, T]$,

$$D_s = \int_{[t,s]} K_u e^{-\int_t^u \rho_r \, dr} d\Theta_u + \delta e^{-\int_t^T \rho_u \, du}$$  

and, taking into account the last trade $\Delta \Theta_T$,

$$D_{T^+} = \int_{[t,T]} K_u e^{-\int_t^u \rho_r \, dr} d\Theta_u + \delta e^{-\int_t^T \rho_u \, du}. \quad (4)$$

For any fixed $t \in [0, T]$, $\delta \geq 0$ and $\kappa > 0$, we define the cost function $J(t, \delta, \cdot, \kappa): \mathcal{A}^\text{cts}_t \to [0, \infty]$ as

$$J(\Theta) := J(t, \delta, \Theta, \kappa) := \mathbb{E}_{t, \delta, \kappa} \left[ \int_{[t,T]} \left( D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s \right],$$

i.e., the expected liquidity cost on the time interval $[t, T]$ when $D_t = \delta$ and $K_t = \kappa$. While we do not exclude the possibility of an infinite cost of a strategy $\Theta \in \mathcal{A}^\text{cts}_t$, it is worth noting that, for any $\Theta \in \mathcal{A}^\text{dis}_t$, the cost is finite due to our standing assumption. Starting with [5] we meet the following notational convention, which will be used throughout the paper: $\mathbb{P}_{t, \kappa}$ is the probability measure under
which the Markov process $K$ starts at time $t$ from $\kappa$, $E_{t,\kappa}$ is the expectation under $P_{t,\kappa}$, and we write $E_{t,\delta,\kappa}$ for the expectation when the expression contains the process $D$ and the starting point at time $t$ in (2) is $\delta$.

Let us briefly recall how the right-hand side of (5) comes into play. Let the best ask price process ($A_s$) be modelled as $A_s = A^0_s + D_s$, where the unaffected best ask price ($A^0_s$) is a càdlàg $\mathcal{H}^1$-martingale. Then, given that the limit order book has the block form, the total cost of a strategy $\Theta \in \mathcal{A}^{cts}_t(x)$ is $\int_{[t,T]} (A_s + \frac{K_s}{2} \Delta \Theta_s) \, d\Theta_s$. A calculation involving integration by parts reveals that the expected total cost equals

$$
E_{t,\delta,\kappa} [A^0_t \Theta_T] + E_{t,\delta,\kappa} \left[ \int_{[t,T]} D_s + \frac{K_s}{2} \Delta \Theta_s \, d\Theta_s \right] = A^0_t x + J(t, \delta, \Theta, \kappa)
$$

with $J(t, \delta, \Theta, \kappa)$ given by (3) (notice that $E_{t,\delta,\kappa} \int_{[t,T]} \Theta_s \, dA_s^0 = 0$ because $A^0_s$ is an $\mathcal{H}^1$-martingale and $\Theta$ is bounded). The first summand in the latter formula describes the expected cost that occurs due to trading in the unaffected price. This cost depends on the strategy $\Theta$, and the term $\frac{K_s}{2} \Delta \Theta_s$ describes the expected liquidity cost, which occurs due to price impact. The second summand in the latter formula describes the expected illiquidity cost, which occurs due to price impact. This cost significantly depends on the strategy and is the object of our study.

Let us now define our value function for continuous trading time $U^{cts} : [0, T] \times [0, \infty)^2 \times (0, \infty) \rightarrow [0, \infty)$ as

$$U^{cts}(t, \delta, x, \kappa) := \inf_{\Theta \in \mathcal{A}^{cts}_t(x)} J(t, \delta, \Theta, \kappa)$$

and the value function for discrete trading time as

$$U^{dis}(t, \delta, x, \kappa) := \inf_{\Theta \in \mathcal{A}^{dis}_t(x)} J(t, \delta, \Theta, \kappa) \geq U^{cts}(t, \delta, x, \kappa).$$

Denoting $\xi_n := \xi_{t_n} = \Delta \Theta_{t_n}$, we can also write the discrete time cost integral as a sum

$$U^{dis}(t, \delta, x, \kappa) = \inf_{\Theta \in \mathcal{A}^{dis}_t(x)} E_{t,\delta,\kappa} \left[ \sum_{t_n \geq t} \left( D_{t_n} + \frac{K_{t_n}}{2} \xi_{t_n} \right) \xi_n \right].$$

Both value functions $U = U^{cts}$ and $U = U^{dis}$ fulfil the boundary conditions

$$U(T, \delta, x, \kappa) = \left( \delta + \frac{\kappa}{2} x \right) x \text{ and } U(t, \delta, 0, \kappa) = 0.$$

Going forward we will use $U$ and $\mathcal{A}_t(x)$ as a notation to indicate that the corresponding statement holds for both the continuous and discrete time case. If a certain statement is referring to only one setting, then we will explicitly use $U^{cts}$ and $\mathcal{A}^{cts}_t(x)$ respectively $U^{dis}$ and $\mathcal{A}^{dis}_t(x)$.

Finally, let us relate the setting in this paper with that in Fruth, Schöneborn, and Urusov (2014). To this end, let the illiquidity coefficient be described by a deterministic strictly positive Borel function $k : [0, T] \rightarrow (0, \infty)$. We introduce the cost and the value functions

$$J_{k(\cdot)}(\Theta) (\equiv J_{k(\cdot)}(t, \delta, \Theta)), \quad U^{cts}_{k(\cdot)}(t, \delta, x), \quad U^{dis}_{k(\cdot)}(t, \delta, x)$$

similarly to (5)–(8) using the illiquidity $k$ in place of $K$. These are the corresponding cost and value functions in Fruth, Schöneborn, and Urusov (2014) (notice that in this case, as $k$ is deterministic, the infima over deterministic and adapted strategies coincide). Again, we will use just the notation $U^{cts}_{k(\cdot)}$ to indicate that the corresponding statement holds for both the continuous and discrete time case. The following lemma is sometimes useful for performing comparisons with the case of deterministically changing illiquidity.
Lemma 3.1 (Stochastic versus deterministic illiquidity).
For all \( t \in [0,T], \delta \geq 0, x \geq 0, \kappa > 0 \), we have
\[
U(t, \delta, x, \kappa) \leq U_{\mathbb{E}, \kappa, [K]}(t, \delta, x).
\]

Proof. \( U(t, \delta, x, \kappa) \) is smaller than or equal to the infimum like the one in (3) respectively (7), but over deterministic strategies. The latter infimum equals \( U_{\mathbb{E}, \kappa, [K]}(t, \delta, x) \) due to (3) and (5).

4 Definition of WR-BR structure

In this section we define WR-BR structure (WR: wait region, BR: buy region) and derive fundamental properties related to it. A detailed introduction of WR-BR structure is provided in Fruth, Schöneborn, and Urusov (2014); we therefore keep the exposition brief in this section.

Before attacking the formal definition of WR-BR structure, we note that the four-dimensional value function \( U \) can be reduced by one dimension due to the following scaling property (its proof is straightforward).

Lemma 4.1 (Optimal strategies scale linearly).
For all \( a \in [0, \infty) \) we have
\[
U(t, a\delta, ax, \kappa) = a^2 U(t, \delta, x, \kappa).
\]

Furthermore, if \( \Theta^* \in \mathcal{A}_t(x) \) is optimal for \( U(t, \delta, x, \kappa) \), then \( a\Theta^* \in \mathcal{A}_t(ax) \) is optimal for \( U(t, a\delta, ax, \kappa) \).

We now define the function \( V : [0, T] \times [0, \infty) \times (0, \infty) \rightarrow [0, \infty) \) by the formula
\[
V(t, y, \kappa) := U(t, 1, y, \kappa).
\]

In what follows we will use the notation \( V^{cts} \) and \( V^{dis} \) where we need to differentiate explicitly between the continuous and the discrete time settings. Notice that for \( \delta > 0 \) we have by Lemma 4.1
\[
U(t, \delta, x, \kappa) = \delta^2 V \left( t, \frac{x}{\delta}, \kappa \right),
\]
that is, the function \( V \) already determines the entire value function \( U \). Technically, the formula above does not yet allow us to draw conclusions about \( U(t, 0, x, \kappa) \) because, for \( \delta = 0 \), the ratio \( y = x/\delta \) is undefined. The claim that \( V \) determines \( U \) is, however, correct without any restriction due to continuity of the value function \( U(t, \cdot, \cdot, \kappa) \) in the pair \((x, \delta)\) (see Proposition A.2).

We first define the buy and wait region and subsequently define the barrier function.

Definition 4.2 (Buy and wait region).
For any \( t \in [0, T] \) and \( \kappa > 0 \), we define the inner buy region as
\[
\text{Br}_{t, \kappa} := \left\{ y \in (0, \infty) \mid \exists \xi \in (0, y) : U(t, 1, y, \kappa) = U(t, 1 + \kappa \xi, y - \xi, \kappa) + \left( 1 + \frac{\kappa}{2} \right) \xi \right\},
\]
and call the following sets the buy region and wait region at time \( t \) for the illiquidity coefficient \( \kappa \):
\[
\text{BR}_{t, \kappa} := \overline{\text{Br}_{t, \kappa}}, \quad \text{WR}_{t, \kappa} := [0, \infty) \setminus \text{Br}_{t, \kappa},
\]
(the bar indicates closure in \( \mathbb{R} \)).

The inner buy region at time \( t \) for illiquidity coefficient \( \kappa \) hence consists of all values \( y \) such that immediate buying at the state \((1, y)\) is value preserving. Due to dynamic programming principle, a non-zero purchase can never create value (i.e., decrease cost), so we always have
\[
U(t, 1, y, \kappa) \leq U(t, 1 + \kappa \xi, y - \xi, \kappa) + \left( 1 + \frac{\kappa}{2} \right) \xi.
\]
The wait region therefore contains all values $y$ such that any non-zero purchase at $(1,y)$ destroys value (i.e., increases cost). Let us note that $Br_{T,\kappa} = (0,\infty)$, $BR_{T,\kappa} = [0,\infty)$ and $WR_{T,\kappa} = \{0\}$. The wait region / buy region conjecture can now be formalized as follows.

**Definition 4.3 (WR-BR structure).**

The value function $U$ has WR-BR structure if there exists a barrier function

$$c: [0, T] \times (0, \infty) \rightarrow [0, \infty]$$

such that for all $t \in [0, T]$ and $\kappa > 0$,

$$BR_{t,\kappa} = (c(t, \kappa), \infty)$$

with the convention $(\infty, \infty) := \emptyset$. For the value function $U^{dis}$ in discrete time to have WR-BR structure, we only consider $t \in \{t_0, \ldots, t_N\}$ and set $c^{dis}(t, \kappa) = \infty$ for $t \notin \{t_0, \ldots, t_N\}$.

It is worth noting that the barrier can be infinite even in continuous time or in discrete time at time points $t_0, \ldots, t_{N-1}$, that is, there can be certain $t$ and $\kappa$, for which it is never optimal to perform a block trade, regardless of how large the remaining position is. We refer to Propositions 5.8 and 5.9 in Fruth, Schöneborn, and Urusov (2014) for sufficient conditions for infinite barrier in the case of deterministically varying $K$.

Let us remark that we always have $c(T, \kappa) = 0$. On the contrary, the barrier is always strictly positive for $t \in [0, T)$ (whenever the value function $U$ has WR-BR structure). The following is a direct generalization of Proposition 5.7 in Fruth, Schöneborn, and Urusov (2014).

**Proposition 4.4 (Wait region near zero).**

Assume that the value function $U$ has WR-BR structure with the barrier $c$. Then, for any $t \in [0, T)$ and $\kappa > 0$, we have $c(t, \kappa) \in (0, \infty]$.

**Proof.** Assume that for some $t \in [0, T)$ and $\kappa > 0$ we have $c(t, \kappa) = 0$. Let us fix some $y > 0$ and define

$$\bar{\xi} := \sup \{ \xi \in (0, y) \mid U(t, 1, y, \kappa) = U(t, 1 + \kappa \xi, y - \xi, \kappa) + \left(1 + \frac{\kappa}{2}\right)\xi \leq y \}.$$

As $U(t, \cdot, \cdot, \kappa)$ is continuous (Proposition A.2), we get

$$U(t, 1, y, \kappa) = U(t, 1 + \kappa \xi, y - \xi, \kappa) + \left(1 + \frac{\kappa}{2}\right)\xi. \quad (12)$$

If $\bar{\xi} < y$, then, due to the scaling property of Lemma A.1 the fact that $(y - \bar{\xi})/(1 + \kappa \bar{\xi}) \in BR_{t,\kappa}$, and the splitting argument of Lemma A.1 we arrive at a contradiction with the definition of $\bar{\xi}$. Thus, $\bar{\xi} = y$, but then formula (12) contradicts Proposition A.3. This completes the proof.

## 5 The WR-BR theorem

In this section we show that the value function exhibits WR-BR structure if $K$ is a diffusion satisfying the following assumption.

**Assumption 5.1.** (Special diffusion).

$K$ is a (possibly time-inhomogeneous) diffusion

$$dK_s = \mu(s, K_s) \, ds + \sigma(s, K_s) \, dW^K_s, \quad K_t = \kappa > 0, \quad (13)$$

for an $(\mathcal{F}_s)$-Brownian motion $W^K$ and $\mu, \sigma: [0, T] \times (0, \infty) \rightarrow \mathbb{R}$ such that, for any initial time $t \in [0, T]$ and starting point $K_t = \kappa > 0$, the stochastic differential equation has a weak solution which is unique in law, is strictly positive and has finite first moments. Furthermore, for all $t \in [0, T]$ and $\kappa > 0$, we have
In Section 8 below we study profitable round trip strategies without assuming that the process \( \eta \) is positive, but we will need part iii) of Assumption 5.1. That is why we write absolute value of \( \eta \) in iii).

**Theorem 5.2.** (WR-BR theorem).

If Assumption 5.1 holds, then there is a unique optimal strategy, and we have WR-BR structure.

In fact, we will see in the proofs that existence and uniqueness of the optimal strategy both in discrete and continuous time as well as WR-BR structure in discrete time hold under parts i)–ii) of Assumption 5.1. We need part iii) only for WR-BR structure in continuous time. Overall, part i) of Assumption 5.1 is the most crucial in the proofs because it is directly linked to the convexity of \( J \). Parts ii) and iii) of Assumption 5.1 are required for more technical aspects of our proofs.

The proof of Theorem 5.2 is given in Appendix B. In Subsections 5.1 and 5.2 we only outline the strategy of the proof. In Lemma 5.3 we show that Assumption 5.1 i)–ii) entails strict convexity of the cost functional \( J \) in the strategy. This is based on the representation

\[
J(\Theta) = \frac{1}{2} E_{t,\delta,\kappa} \left[ \frac{D_{T+}^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s D_s^2 ds \right] 
\]

(see the proof of Lemma 5.3 in Appendix B), which also proves to be useful in the study of profitable round trip strategies in Section 8. The strict convexity of \( J \) in turn guarantees existence and uniqueness of the optimal strategy. As we show in Subsection 5.2, the uniqueness excludes WR-BR-WR and other situations: at any upper boundary of a buy region it must be equally optimal to wait and to execute the strictly positive trade to the lower boundary of the buy region. We first pursue this line of argument for the discrete time case and then transfer it to continuous time and thus do not use the Hamilton-Jacobi-Bellman equation.

Theorem 5.2 does not cover all models which result in WR-BR structure. E.g., restricting trading to only two points in time always yields WR-BR structure irrespective of Assumption 5.1, see Proposition 7.1. Furthermore, in the case of deterministically varying \( K \) we always have WR-BR structure in discrete time and, for continuous \( K \), in continuous time, see Fruth, Schöneborn, and Urusov (2014).

In Section 6 we show that Assumption 5.1 and thus WR-BR structure hold in several standard models. In Section 7 we provide examples violating WR-BR structure, which highlights that some assumptions on \( K \) are necessary for it. As we will see in Section 8, part i) of Assumption 5.1 is also related to the absence of profitable round trip trading strategies in a zero spread two-sided order book model.

### 5.1 Existence of a unique optimal strategy

**Lemma 5.3.** (Costs are convex in the strategy).

Let Assumption 5.1 i)–ii) hold. Then, for all \( t \in [0,T] \), \( \delta \geq 0 \) and \( \kappa > 0 \), the function \( J(\cdot) \equiv J(t, \delta, \cdot, \kappa) \) is finite and strictly convex on \( \mathcal{A}_t \).

The strict convexity of \( J \) guarantees the uniqueness of an optimal strategy provided it exists. Moreover, we can use the convexity together with a Komlós-type argument to get the existence of an optimal strategy both in discrete and in continuous time:
Proposition 5.4. (Existence and uniqueness of an optimal strategy).
Let Assumption 5.1 i)--ii) hold. Then, for all \( t \in [0,T], \delta \geq 0, x \geq 0 \) and \( \kappa > 0 \), there exists a unique optimal strategy, i.e., there exists a unique \( \Theta^* = \Theta^*(t, \delta, x, \kappa) \in \mathcal{A}_t(x) \) with
\[
J(t, \delta, \Theta^*, \kappa) = \inf_{\Theta \in \mathcal{A}_t(x)} J(t, \delta, \Theta, \kappa).
\]

5.2 Wait and buy region structure

In order to prove WR-BR structure, we first exploit the existence and uniqueness of Proposition 5.4 together with a technical result (Proposition B.1) in order to get WR-BR structure in discrete time:

Proposition 5.5. (Discrete time: WR-BR structure).
Let Assumption 5.1 i)--ii) hold. Then the value function \( U_{\text{dis}} \) has WR-BR structure.

The line of argument used in the proof of Proposition 5.5 does not extend directly to continuous time. We, therefore, transfer the discrete time result to continuous time using approximation techniques (Lemmas B.2 and B.3), and this is where we need part iii) of Assumption 5.1.

Proposition 5.6. (Continuous time: WR-BR structure).
Let Assumption 5.1 i)--iii) hold. Then the value function \( U_{\text{cts}} \) has WR-BR structure.

6 Example models with WR-BR structure

6.1 Analytical results

By Theorem 5.2 any model satisfying Assumption 5.1 has WR-BR structure. In this section, we show that Assumption 5.1 is satisfied by several standard processes. We start with a deterministic \( K \).

Proposition 6.1. (Deterministic case).
Assume that \( K: [0,T] \rightarrow (0, \infty) \) is deterministic and two times continuously differentiable, \( \rho: [0,T] \rightarrow (0, \infty) \) is continuously differentiable with \( K_t^2 + 2\rho_t K_t > 0 \) for all \( t \in [0,T] \). Then Assumption 5.1 holds, and the value function has WR-BR structure.

Proof. Condition i) is equivalent to \( K_t^2 + 2\rho_t K_t > 0 \), and ii), iii) are clearly satisfied for deterministic continuous \( K \).

We remark that Proposition 6.1 is not the best result one can get for a deterministic \( K \). In Fruth, Schöneborn, and Urusov (2014) we prove that we have WR-BR structure whenever \( K \) is continuous and deterministic.

Let us now turn to a time-homogeneous geometric Brownian motion (GBM). Notice that, due to the homogeneity in time, it is enough to verify the conditions in Assumption 5.1 only under measures \( \mathbb{P}_{0,\kappa} \).

Proposition 6.2. (GBM case).
Let \( K \) be a geometric Brownian motion
\[
dK_t = \mu K_t dt + \sigma K_t dW_t^K, \quad K_0 = \kappa > 0.
\]
Consider a constant resilience \( \rho_t \equiv \bar{\rho} > 0 \) such that \( 2\bar{\rho} + \bar{\mu} - \bar{\sigma}^2 > 0 \). Then Assumption 5.1 holds, and the value function has WR-BR structure.
The proof of this proposition is given in Appendix C. Also see Fruth (2011) for alternative conditions ensuring WR-BR structure in the GBM case.

At this point it is worth noting (see Fruth (2011)) that the barrier $c(t,\kappa)$ in the GBM case has the form $c(t)/\kappa$. This is not desirable from the practical viewpoint because this prescribes to trade more aggressively when $\kappa$ is large. The intuition behind this effect is as follows. Even when the starting point $\kappa$ of the process $K$ is large, we do not have an incentive to wait because under the GBM model $K$ is going to remain large for some time. On the other hand, we profit from resilience while trading (in other words, we profit from the upcoming limit orders that arrive in place of the consumed ones with the rate which is the resilience).

The natural generalisation of the GBM model is the CEV model (e.g., see Davydov and Linetsky (2001)), which is related to the squared Bessel process (e.g., see Carr and Linetsky (2006)). In that model the hyperbolic form $c(t)/\kappa$ for the barrier does not hold any longer. However, the previously developed intuition suggests to try a process with mean reversion. That is, a behaviour, where the process $K$ is going to be quickly reduced whenever the starting point $\kappa$ of $K$ is big, looks particularly appealing from the economic point of view. The natural model is then, speaking informally, a “squared Bessel process with mean reversion”, which is the Cox-Ingersoll-Ross (CIR) process.

**Proposition 6.3.** (CIR case)

Let $K$ be a Cox-Ingersoll-Ross process

$$dK_t = \bar{\mu}(\bar{K} - K_t) \, dt + \bar{\sigma} \sqrt{K_t} \, dW^K_t, \quad K_0 = \kappa > 0,$$

where $\bar{K}, \bar{\mu}, \bar{\sigma} > 0$. Consider a constant resilience $\rho_t \equiv \bar{\rho} > 0$ such that

$$2\bar{\rho} \geq \bar{\mu} > 2\bar{\sigma}^2/\bar{K}.$$

Then Assumption 5.1 holds, and the value function has WR-BR structure.

The proof of this proposition is presented in Appendix C.

### 6.2 Numerical results for CIR examples

To illustrate the WR-BR structure and the corresponding optimal strategy, we present the result of a numerical implementation of two specific parameter sets for the CIR process in this subsection.

One can use the Markov chain approximation method introduced by Kushner and his co-authors, e.g., see Kushner and Dupuis (2001), to implement our optimization problem for the mean-reverting CIR process as introduced in Proposition 6.3. For a formal convergence proof and a detailed explanation of this numerical scheme see Fruth (2011), Chapter 3. For a numerical implementation of the CIR example one needs to specify nine parameters:

$$\bar{\rho}, \bar{\mu}, \bar{K}, \bar{\sigma}, \quad T, y_{\max}, \kappa_{\max}, \quad \Delta t, h.$$

The first four parameters specify the resilience speed and the parameters for the stochastic liquidity CIR process (mean-reversion speed, mean-reversion level and volatility). $[0, T] \times [0, y_{\max}] \times [0, \kappa_{\max}]$ is the space, where we approximate the dimension-reduced value function $V$ introduced in (11). $\Delta t$ and $h$ correspond to the grid size in time and space respectively. It is crucial that $\Delta t$ is significantly smaller than $h$ for the numerical scheme to yield reasonable results.

In Figure 4 we state the result of our implementation of the CIR process for two different values of mean-reversion speed $\bar{\mu}$. The size of the wait region is decreasing in time in both cases (see the plots on the top), i.e., buying becomes more aggressive as the investor runs out of time. This is also the case for constant liquidity and this is what we intuitively expect. However, examples with the barrier
being increasing in time can also be constructed, see Figure 6.1 in Fruth, Schöneborn, and Urusov (2014).

The barrier profile \( c(t, \kappa) \) is increasing in \( \kappa \) for the high mean reversion case\footnote{This drives how the optimal strategy \( \Theta_t(\omega) \) reacts to the illiquidity process \( K_t(\omega) \): At times of growing \( K_t \) it falls behind the constant liquidity strategy, while trading is accelerated when \( K_t \) decreases. In analogy to the terminology of aggressive in the money execution introduced in Schied and Schöneborn (2009), such an execution strategy could be called aggressive in the liquidity. For the low mean reversion case, the barrier is decreasing in \( \kappa \), resulting in the opposite relationship (passive in the liquidity).

In order to understand the difference between the barriers for low and high mean reversion in more detail, let us first consider the case of constant \( K = \kappa \) (i.e., the model of Obizhaeva and Wang (2013) with constant, deterministic liquidity). If \( \kappa \) is large, then the market is easily dislocated, and we have to accept large dislocations \( \delta \) and hence have a small boundary \( c(t, \kappa) \). If on the other hand \( \kappa \) is small then we can maintain a smaller dislocation and hence larger \( c \). The barrier \( c(t, \kappa) \) is hence decreasing in \( \kappa \). This effect intuitively carries over to CIR processes with small mean reversion because \( K \) on average will not change significantly during the execution time horizon. However if the mean reversion of the CIR process is large, then we do expect a change of \( K \) over time depending on the current level of \( K_t = \kappa \): If \( \kappa \) is large and the mean reversion is high, then it is better to wait because the process \( K \) is going to quickly shrink to the smaller level \( \bar{K} \) which will allow cheaper execution. Respectively if \( \kappa \) is small then we expect \( K \) to grow, resulting in more expensive execution in the future and hence an incentive to speed up execution now. The net result is a barrier \( c(t, \kappa) \) that is increasing in \( \kappa \).

In the extreme case, we obtain an infinite barrier for large \( \kappa \). A straightforward generalisation of Proposition 5.9 in Fruth, Schöneborn, and Urusov (2014) gives us the following sufficient condition for infinite barrier. If, for some \( 0 \leq t_1 < t_2 \leq T \) and \( \kappa > 0 \), the function \( t \mapsto \mathbb{E}_{t_1, \kappa} K_t \) is continuous on \([t_1, t_2]\) and

\[
\kappa e^{-\int_{t_1}^{t_2} \rho_u \, du} > \mathbb{E}_{t_1, \kappa} K_{t_2}
\]

(intuition: the increase in liquidity outweighs the resilience), then \( Br_{t_1, \kappa} = \emptyset \), i.e., \( c(t_1, \kappa) = \infty \).

Applying this statement in the CIR case and using the well-known formula

\[
\mathbb{E}_{t, \kappa} K_s = \bar{K} + e^{-\bar{\rho}(s-t)}(\kappa - \bar{K}) \quad \text{for} \ s \geq t,
\]

we obtain that, for \( \bar{\mu} > \bar{\rho} \), we have infinite barrier for sufficiently large values of \( \kappa \), which is what we see on the top right plot in Figure\footnote{The apparent lack of smoothness in the corresponding plot in Figure\footnote{is an artifact of the discretisation scheme that we used rather than a property of the actual barrier.} is an artifact of the discretisation scheme that we used rather than a property of the actual barrier.}.

There are flat stretches in the optimal strategy (see both plots in the bottom of Figure\footnote{we will see counterintuitive examples of WR-BR-WR structure. In such WR-BR-WR situations and \( \delta = 0 \) we would not get the typical initial discrete trade in the optimal strategy, but the optimal strategy would start with a flat stretch.} the effect is more pronounced in the plot on the right). During these time spans, the optimal ratio of outstanding shares over order book deviation is in the interior of the wait region. As opposed to the constant liquidity case (the dotted line in the bottom plots), the optimal strategy displays a stop-and-go pattern.

In Section\footnote{we provide a couple of examples that do not follow the WR-BR structure. We find that cases of WR-BR-WR structure can occur: when a large number of shares remains to be purchased, it is optimal to wait, while buying is optimal if a smaller number of shares is remaining. We first} we will see counterintuitive examples of WR-BR-WR structure. In such WR-BR-WR situations and \( \delta = 0 \) we would not get the typical initial discrete trade in the optimal strategy, but the optimal strategy would start with a flat stretch.
Figure 1: Implementation for the CIR process with $\bar{\rho} = 5$, $\bar{K} = 1$, $\sigma^2 = 2$, $T = 0.25$, $y_{\text{max}} = 30$, $\kappa_{\text{max}} = 2$, $\Delta t = 0.00005$, $h = 0.05$. Left: $\bar{\mu} = 3$. Right: $\bar{\mu} = 10$. On the top, the barriers $\kappa \mapsto c(t,\kappa)$ are plotted for $t \in \{0, \frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T\}$ (top-down). The Euler scheme is used to simulate a path $K_t(\omega)$ of the CIR process. In the middle plots, $K_t(\omega)$ is compared with the mean-reverting level $\bar{K} = 1$ (dotted). In the bottom plots, the optimal strategy for $\delta = 0$, $x = 100$ and the simulated path $K_t(\omega)$ is compared to the constant liquidity strategy (dotted).
present a two-scenario model with WR-BR-WR structure and discuss it in Subsection 7.1 in detail. This discussion sharpens intuition about the reasons of such a behaviour and provides an intuitive explanation why this phenomenon does not appear with deterministic liquidity dynamics. Using this intuition Fruth (2011) provides a WR-BR-WR example in a certain two scenario continuous-time model.

Furthermore, the discussion in Subsection 7.1 inspires certain parameter choices in the CIR model that might also lead to WR-BR counterexamples. In Subsection 7.2 we perform a numerical check for one such parameter choice in the CIR model in discrete time and indeed recover another WR-BR counterexample, now in the CIR framework. Our examples are in discrete time with trading occurring at three points in time. The following proposition shows that WR-BR structure always holds if trading occurs at two time points only.

**Proposition 7.1.** (WR-BR structure for two trading instances).

Let \( N = 1 \), i.e., \( 0=t_0 < t_1 = T \), and denote \( a_0 := e^{-\int_{t_0}^{t_1} \rho_s \, ds} \). Then the value function has WR-BR structure with

\[
V^{\text{dis}}(t_0, y, \kappa) = \begin{cases} 
\frac{1}{2} \mathbb{E}_{t_0, \kappa}[K_T] y^2 + a_0 y + \left\{ \frac{[(\mathbb{E}_{t_0, \kappa}[K_T] - \kappa a_0)]y - (1 - a_0)^2}{2\kappa + 2\mathbb{E}_{t_0, \kappa}[K_T] - \kappa a_0} \right\} & \text{if } y > c(t_0, \kappa) \\
0 & \text{if } y < c(t_0, \kappa) \\
ie & \text{otherwise}
\end{cases}
\]

\[
c(t_0, \kappa) = \begin{cases} 
\frac{1}{\mathbb{E}_{t_0, \kappa}[K_T] - \kappa a_0} & \text{if } \mathbb{E}_{t_0, \kappa}[K_T] > \kappa a_0 \\
\infty & \text{otherwise}
\end{cases}
\]

**Proof.** We know that \( U^{\text{dis}}(t_1, \delta, x, \kappa) = (\delta + \frac{x}{2}) x \). The assertion follows from

\[
U^{\text{dis}}(t_0, \delta, x, \kappa) = \min_{\xi \in [0, x]} \left\{ \left( \delta + \frac{\kappa \xi}{2} \right) \xi + \mathbb{E}_{t_0, \kappa}\left[ U^{\text{dis}}(t_1, (\delta + \kappa \xi)a_0, x - \xi, K_T) \right] \right\}
\]

Note that we have not made any specific assumptions on the distribution of \( K_T \) in Proposition 7.1

### 7.1 A two scenario WR-BR counterexample

Let us assume that the process \( K \) is not driven by a diffusion, but instead is given by a finite number of scenarios. The case of a single scenario implies a deterministic evolution of \( K \) which always results in a WR-BR structure. We therefore focus on the second simplest case of two equally likely scenarios \( A \) and \( B \), i.e., \( \Omega = \{\omega_A, \omega_B\} \), \( \mathbb{P}(\{\omega_A\}) = \mathbb{P}(\{\omega_B\}) = 1/2 \), and consider three trading instances \( \{t_0, t_1, t_2\} \), i.e., we consider a discrete time example with \( N = 2 \). To fully specify this two scenario model, we need to choose seven constants

\[
a_0 := e^{-\int_{t_0}^{t_1} \rho_s \, ds}, \quad a_1 := e^{-\int_{t_1}^{t_2} \rho_s \, ds}, \quad \kappa_0 := K_{t_0}, \quad \kappa_1 := K_{t_1}(\omega_A), \quad \kappa_2 := K_{t_1}(\omega_B), \quad \kappa_1^B := K_{t_2}(\omega_A), \quad \kappa_2^B := K_{t_2}(\omega_B).
\]

**Proposition 7.2.** With the parameter values given in Figure 4, we obtain WR-BR-WR structure, i.e., there are two threshold values \( 0 < c_1 < c_u < \infty \) such that the buy region at time \( t_0 \) is given by \( B_{t_0} = (c_1, c_u) \).

A formal proof of Proposition 7.2 is given in Appendix D. We now present an intuitive explanation of the phenomenon described here, which will inspire the construction of a WR-BR counterexample in the CIR model presented in Subsection 7.2.

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How to compute \( c_t \) and \( c_u \) is explained in the proof. For the discussion below we notice that \( c_t = 0.95 \) and \( c_u = 5.75 \). For a trade \( \xi \), we define

\[
\tilde{U}^{\text{dis}}(t_0, \delta, x, \kappa_0; \xi) := \left( \delta + \frac{\kappa_0}{2} \xi \right) \xi + \mathbb{E} \left[ U^{\text{dis}}(t_1, (\delta + \kappa_0 \xi) a_0, x - \xi, \xi_1) \right],
\]

which is a cost of buying \( \xi \) shares at time \( t_0 \) and optimal trading thereafter. It is easy to see that \( \tilde{U}^{\text{dis}} \) is piecewise quadratic in \( \xi \).

To illustrate the dynamics of the optimal strategy, we take different \( x \) and plot

\[
\xi \mapsto \tilde{U}^{\text{dis}}(t_0, 1, x, 1.95; \xi)
\]

in Figure 3. When the total order is as small as \( x = 0.9 \), it is optimal not to do an initial trade. The transition from wait to buy region is approximately at \( x = 0.95 \). For \( x = 1 \), we are in the buy region and one optimally trades about two percent of the total order at time \( t_0 \). But at \( x = 5.75 \), we switch from buy to wait region and stay in the wait region for all larger values of \( x \). The graph for \( x = 5.75 \) illustrates the non-uniqueness of the optimal strategy at the transition from buy to wait region.

Intuition might suggest that the larger the remaining position \( x \) at time \( t_0 \), the larger the initial trade \( \xi_0 \). The downside of trading at time \( t_0 \) is that the full initial impact \( \delta \) is influencing the cost functional (at later points in time this initial impact is partially decayed already). The upside is a more balanced distribution of new impact across an additional time point (any impact generated at time \( t_0 \) will already be partially decayed at time \( t_1 \)). These two effects are the only drivers in the case of deterministic \( K \), and the second effect grows faster in the remaining position \( x \) than the first effect.

If \( K \) evolves stochastically, then a third effect comes into play: trading at times after \( t_0 \) can respond to new information gained about \( K \) (such as whether scenario \( A \) or \( B \) occurred). This effect can dominate the second effect for large remaining positions \( x \).

Let us now analyze the situation for different values of \( \kappa_0 \) while keeping the other model parameters including \( \kappa_2^A, \kappa_1^B, \kappa_2^B \) and \( \kappa_2^B \) fixed. Figure 4 indicates for each point \((\kappa_0, x)\) if it belongs to the buy or wait region. It is created by computing the optimal initial trade \( \xi(\kappa_0, x) \) of \( \tilde{U}^{\text{dis}}(t_0, 1, x, \kappa_0; \xi) \) analytically. WR-BR-WR structure occurs for \( \kappa_0 \in (1.94, 2) \). The upper barrier from buy to wait region has an asymptote at \( \kappa_0 = 1.94 \). For the case \( \kappa_0 = 1.95 \) that we discussed in Figure 3 the small dots on the right-hand side of Figure 4 point out the transitions from wait to buy region and buy to wait region respectively. For expensive \( \kappa_0 \geq 2 \), we are not trading irrespectively of the size of the total order. For inexpensive \( \kappa_0 \leq 1.94 \), we have the usual WR-BR situation. On the interval in between, the large investor has an incentive not to trade for large positions \( x \). The resilience between \( t_0 \) and \( t_1 \) is extremely low and waiting until \( t_1 \) has the advantage of gaining information whether scenario \( A \) or \( B \) has occurred. That is there is a tradeoff between gaining information by waiting until the next time instance and attracting resilience by trading right now.
Figure 3: For the parameters from Figure 2 and total order size $x = 0.9, 1, 5.75, 20$, the graphs plot the dependence of the costs $\tilde{U}_{dis}(t_0, 1, x, 1.95; \xi)$ on the initial trade $\xi$.

Figure 4: For the parameters from Figure 2 but different values of $\kappa_0$, we illustrate the wait and buy region. Looking more closely at the large dot $(\kappa_0, x) = (2, 1)$ yields the picture on the right-hand side. The buy region has the shape of a wedge.
Informally, one can view the two scenario model of this section as an approximation of a diffusive model that has almost half of its scenarios close to scenario A and another half close to scenario B. One can therefore expect that WR-BR structure is lost in such diffusive models.

7.2 Cox-Ingersoll-Ross process in discrete time

In Section 6, we have considered examples of diffusive models and shown that they have WR-BR structure if certain conditions are met. For the case of the CIR process

\[ dK_s = \mu (\bar{K} - K_s) ds + \sigma \sqrt{K_s} dW^K_s, \]

let us now consider three trading times \( \{t_0, t_1, t_2\} \) with

\[ t_0 = 0, \quad t_1 = 0.0072, \quad t_2 = 1.0072, \quad \bar{\rho} \equiv 1.3863, \]
\[ \bar{\mu} = 0.6931, \quad \bar{K} = 1, \quad \bar{\sigma} = 5.2523. \]

This example violates the conditions of Proposition 6.3. It is inspired by the two scenario model presented in Subsection 7.1. E.g., \( t_1 \) is close to \( t_0 \), and the high volatility makes illiquid scenarios with \( K_t \) being very different from \( \bar{K} \) likely to occur.

Using Proposition 7.1 and the density function of the CIR process together with a numerical integration scheme, we can compute \( \bar{U}^{d_{\xi}}(t_0, 1, x, \kappa_0; \xi) \) from dynamic programming. For each point \( (\kappa_0, x) \), we can calculate the costs of different trades \( \xi \) from an equidistant grid \( \{0, d\xi, ..., x\} \). We can then infer that the point \( (\kappa_0, x) \) belongs to the wait region if the costs for \( \xi = 0 \) are smaller than the costs on the remaining grid.

Executing this scheme for several points \( (\kappa_0, x) \) yields Figure 5. As for the two scenario model, there exist choices of \( \kappa_0 \) that lead to WR-BR-WR structure. But instead of a wedge-shaped buy region, we get a tongue-shaped upper wait region, which is located around the mean-reversion level \( \bar{K} = 1 \).

![Figure 5](image)

**Figure 5:** This figure shows a WR-BR-WR example for the CIR process with parameters (16) and three trading instances. Points \( (\kappa_0, x) \in \{0.1, 0.2, ..., 2.1\} \times \{0.2, 0.4, ..., 8\} \) are considered. The wait region is shaded black.

\(^3\)See Fruth (2011) for a WR-BR-WR example for the time-inhomogeneous GBM and three trading instances.
8 Profitable round trip strategies

So far we have only considered one side of the limit order book. In this section, we extend our model and include the other side of the limit order book. In such two-sided limit order books, round trip strategies are possible and we determine under which conditions they can be profitable.

Without loss of generality we now consider the starting time 0. We model strategies that both buy and sell the asset as a pair \((\Theta, \hat{\Theta})\), where \(\Theta \in A_0\) and \(\hat{\Theta} \in A_0\) describe the number of shares which the investor bought respectively sold starting from time 0. The position at time \(t\) is given by \(\Theta_t - \hat{\Theta}_t\), and a round trip strategy is characterised by \(\Theta_{T+} - \hat{\Theta}_{T+}\). Recall that, by definition of \(A_0\), \(\Theta_{T+}\) and \(\hat{\Theta}_{T+}\) are bounded random variables. If \(A_t\) and \(B_t\) are the best ask and best bid prices respectively, then the total cost of a strategy \((\Theta, \hat{\Theta})\) is given by

\[
C(\Theta, \hat{\Theta}) := \int_{[0,T]} \left( A_t + \frac{K_t}{2} \Delta \Theta_t \right) d\Theta_t - \int_{[0,T]} \left( B_t - \frac{K_t}{2} \Delta \hat{\Theta}_t \right) d\hat{\Theta}_t. \tag{17}
\]

We now present two different models for two-sided limit order books. The corresponding models for deterministic \(K\) are discussed in Fruth, Schöneborn, and Urusov (2014). First, we consider a two-sided limit order book with bid-ask spread that depends on trading activity.

**Model 8.1.** (Dynamic spread model).

The best ask and best bid price processes \(A\) and \(B\) in (17) are modelled as \(A_t := A^a_t + D_t\) and \(B_t := B^b_t - E_t\), where the unaffected best ask and best bid price processes \(A^u_t\) and \(B^u_t\) are càdlàg \(H^1\)-martingales with \(B^u_t \leq A^u_t\) for all \(t \in [0, T]\), and

\[
D_t := D_0 e^{-\int_0^t \rho_s dS_s} + \int_{[0,t]} K_s e^{-\int_s^t \rho_u dS_u} d\Theta_s, \quad t \in [0, T+], \tag{18}
\]

\[
E_t := E_0 e^{-\int_0^t \rho_s dS_s} + \int_{[0,t]} K_s e^{-\int_s^t \rho_u dS_u} d\hat{\Theta}_s, \quad t \in [0, T+], \tag{19}
\]

with some given non-negative initial values \(D_0 \geq 0\) and \(E_0 \geq 0\).

**Proposition 8.2.** (Profitable round trips in the dynamic spread model).

In the dynamic spread model round trip trading strategies cannot be profitable. That is, for all \(\kappa > 0\), \(D_0 \geq 0\) and \(E_0 \geq 0\), for all admissible \((\Theta, \hat{\Theta})\) with \(\Theta_{T+} - \hat{\Theta}_{T+} = x > 0\), there is an admissible \(\Theta\) with \(\Theta_{T+} = x\) such that \(E_{0,\kappa}[C(\Theta, \hat{\Theta})] \geq E_{0,\kappa}[C(\Theta, 0)]\); also the symmetric statement with \(x < 0\) holds true.

Furthermore, the expected execution costs of a buy (or sell) program that builds up a deterministic position of say \(x \in \mathbb{R}\) shares cannot be decreased by intermediate sell (resp. buy) trades. That is, for all \(\kappa > 0\), \(D_0 \geq 0\) and \(E_0 \geq 0\), for any admissible \((\Theta, \hat{\Theta})\) with \(\Theta_{T+} - \hat{\Theta}_{T+} = x > 0\), there is an admissible \(\Theta\) with \(\Theta_{T+} = x\) such that \(E_{0,\kappa}[C(\Theta, \hat{\Theta})] \geq E_{0,\kappa}[C(\Theta, 0)]\).

We omit the proof because it is a direct extension of the corresponding Proposition 3.4 in Fruth, Schöneborn, and Urusov (2014). Let us now consider an alternative model for a two-sided limit order book in which the spread is constantly zero.

**Model 8.3.** (Zero spread model).

The best ask and best bid price processes in (17) are modelled as \(A^\dagger_t := B^\dagger_t := S^u_t + D^\dagger_t\), where the unaffected price \(S^u_t\) is a càdlàg \(H^1\)-martingale, and

\[
D^\dagger_t := D_0^e e^{-\int_0^t \rho_s dS_s} + \int_{[0,t]} K_s e^{-\int_s^t \rho_u dS_u} (d\Theta_s - d\hat{\Theta}_s), \quad t \in [0, T+], \tag{20}
\]

with some given initial value \(D_0^e \in \mathbb{R}\).
There is a subtle difference in understanding price manipulation between the dynamic and zero spread models. In the discussion of profitable round trip strategies in the dynamic spread model (see Proposition 8.2) we considered arbitrary initial values $D_0 \geq 0$ and $E_0 \geq 0$ in (18) and (19). In contrast to this, in the discussion of profitable round trip strategies in the zero spread model (Theorem 8.3 which follows) we will consider $D_0 = 0$ in (20). Whenever $D_0 \neq 0$ we usually have profitable round trip strategies in the zero spread model, and this is due not to properties of the model, but rather to the fact that both buy and sell orders are executed at the same price therefore, profitable round trips will make use of the initial deviation $D_0^\uparrow$ from the unaffected price $S^u$ and of the fact that, due to the resilience, the absolute value of this deviation decreases to zero in the absence of trading.

In order to study profitable round trip strategies in the zero spread model let us introduce the notations

$$\Theta^\downarrow := \Theta - \tilde{\Theta}$$

(21)

for the composite strategy, which includes both buy and sell orders, and, by analogy with (5),

$$J^\downarrow(t, \Theta^\downarrow, \kappa) := J^\downarrow(t, \delta, \Theta^\downarrow, \kappa) := \mathbb{E}_{t, \delta, \kappa} \left[ \int_{[t,T]} \left( D_s^\downarrow + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s \right]$$

(22)

for the cost function. As in (5), the subscript in $\mathbb{E}_{t, \delta, \kappa}$ means that we start at time $t$ with $D_t^\uparrow = \delta$ and $K_t = \kappa$. The precise explanation of how (22) comes into play is similar to the explanation of (5) given in the paragraph following the one that contains (5). Namely, consider a strategy $(\Theta, \tilde{\Theta}) \in \mathcal{A}_t \times \mathcal{A}_t$ that acquires $\Theta^\downarrow_{T^+} = z$ shares on the time interval $[t, T]$ ($x \in \mathbb{R}$ is deterministic). The total cost of this strategy is (cf. (17))

$$\int_{[t,T]} \left( S^u_s + D_s^\downarrow + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s - \int_{[t,T]} \left( S^u_s + D_s^\downarrow - \frac{K_s}{2} \Delta \tilde{\Theta}_s \right) d\tilde{\Theta}_s.$$

A calculation involving integration by parts and using that $S^u$ is an $\mathcal{H}^1$-martingale as well as that $\Theta$ and $\tilde{\Theta}$ are bounded reveals that the expected total cost equals

$$S^u_t + \mathbb{E}_{t, \delta, \kappa} \left[ \int_{[t,T]} \left( D_s^\downarrow + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s - \int_{[t,T]} \left( D_s^\downarrow - \frac{K_s}{2} \Delta \tilde{\Theta}_s \right) d\tilde{\Theta}_s \right].$$

Again, the first summand, which is trivial and moreover vanishes for round trip strategies, describes the expected cost that occurs due to trading in the unaffected price. The second summand in the latter formula, which describes the expected liquidity cost, is in general larger than $J^\uparrow(t, \delta, \Theta^\uparrow, \kappa)$, but it is equal to $J^\downarrow(t, \delta, \Theta^\downarrow, \kappa)$ whenever $\Theta_{T^+} = \tilde{\Theta}_{T^+}$ equals the variation of $\Theta^\downarrow$ over $[t, T]$. It remains to notice that the latter can always be assumed without loss of generality (and, moreover, it does not make sense economically to consider strategies $(\Theta, \tilde{\Theta})$ with $\Theta_{T^+} = \tilde{\Theta}_{T^+}$ being strictly greater than the variation of $\Theta^\downarrow$ over $[t, T]$ because this means that buying and selling happen simultaneously).

---

$^4$If the spread between the bid and ask is a positive constant with trades dislocating both bid and ask prices by the same amount then profitable round trip strategies are possible for sufficiently large $D_0^\uparrow$.

$^5$To explain this point in more detail, we assume that $D_0^\uparrow \neq 0$, consider some $\kappa > 0$ and arbitrary $\epsilon \in (0, T]$ and choose

$$z = \frac{1 - e^{-\int_0^\epsilon \rho_u \, du}}{E_0, K_s + \kappa (1 - e^{-\int_0^\epsilon \rho_u \, du})}.$$

Now consider the strategy (using notation (21))

$$\Theta^\downarrow_s = -zD_0^\uparrow I_{(0,\epsilon)}(s), \quad s \in [0, T^+].$$

A straightforward calculation reveals that the expected cost is

$$\frac{1}{2} z^2 (D_0^\uparrow)^2 \left\{ (-2 - \kappa z) + (2 - 2\kappa z) e^{-\int_0^\epsilon \rho_u \, du} + z E_0, K_s \right\} < \frac{1}{2} z^2 (D_0^\uparrow)^2 \left\{ (-2 - \kappa z) \left( 1 - e^{-\int_0^\epsilon \rho_u \, du} \right) + z E_0, K_s \right\} < 0,$$

that is, $(\Theta, \tilde{\Theta})$ is a profitable round trip strategy.
We write $J^T_T$ in \[ (22) \] with the subscript $T$ to emphasize the time horizon explicitly. We consider diffusion setting \[ (13) \] for all finite time horizons $T < \infty$ and introduce the function $\eta: \mathbb{R}_+ \times (0, \infty) \to \mathbb{R}$ by the formula

$$ \eta(s, \kappa) := \frac{2\rho_s}{\kappa} + \frac{\mu(s, \kappa)}{\kappa^2} - \frac{\sigma^2(s, \kappa)}{\kappa^3}, $$

that is, we have $\eta_s = \eta(s, K_s)$ for $\eta_s$ as in Assumption \[ (5.1) \].

**Theorem 8.4.** (Profitable round trips in the zero spread model).

In the zero spread model suppose that Assumption \[ (5.1) \] ii) holds for all finite $T < \infty$ and

(A) the resilience is bounded away from zero ($\rho_s \geq \bar{\rho} > 0$) as well as, for all $t \geq 0$ and $\kappa > 0$, the function $s \mapsto \mathbb{E}_{t, \kappa}[K_s]$, $s \in [t, \infty)$, is bounded.

We then have the following classification:

1. If $\eta \geq 0$ everywhere, then all round trip strategies starting at any time $t \geq 0$ with $D^T_t = 0$ have nonnegative costs.

2. Under Assumption \[ (5.1) \] iii), if $\eta < 0$ in some $[t, t + \Delta t] \times [\kappa - \epsilon, \kappa + \epsilon]$, then there are profitable round trip strategies starting at $t$ with $D^T_t = 0$.

Assumption (A) is satisfied for a wide range of processes $K$ including stationary processes such as the CIR process as well as the GBM process with non-positive drift ($\mu \leq 0$ in \[ (15) \]) whenever the resilience is bounded away from zero.

We will see in the proof of Theorem \[ (8.4) \] in Appendix E that the role of Assumption (A) is to ensure that liquidation of a random but bounded position that the investor has at some time $t + \Delta t$ can be achieved for arbitrarily small cost if $D^T_{t+\Delta t} = 0$ and the time horizon $T$ is large. The latter property seems natural to expect in reasonable models when the resilience is bounded away from zero. In fact, one might replace Assumption (A) with any other assumption that ensures the property stated above.

Parts ii) and iii) of Assumption \[ (5.1) \] are required for some technical aspects of our proof. What is instrumental there is the generalization of formula \[ (14) \] to the case of not necessarily positive $\eta$. The main message of Theorem \[ (8.4) \] can be somewhat loosely described as follows: if $\eta \geq 0$ everywhere, then the cost functional $J$ is convex in the strategy and round trips cannot be profitable; if $\eta < 0$ somewhere, then the cost functional is not convex and profitable round trip strategies exist.

The results of this section reveal a link between the models for two-sided limit order books in our present setting (stochastic diffusive $K$): If Assumption \[ (5.1) \] i) holds, then optimal strategies in the dynamic spread model are of WR-BR structure and profitable round trip strategies do not exist in the zero spread model. If Assumption \[ (5.1) \] i) is violated, then optimal strategies in the dynamic spread model do not need to be of WR-BR structure, and round trip strategies in the zero spread model do not need to result in costs.

If only deterministic trading strategies are considered, then only the expected evolution of $K$ matters and, in the case $\mu(s, \kappa)$ is affine in $\kappa$, $\bar{\eta}_s := \frac{2\rho_s}{\kappa} + \frac{\mu(s, K_s)}{\kappa^2} \geq 0$ is sufficient to prevent free (or even profitable) deterministic round trip strategies. As $\eta_s = \bar{\eta}_s - \frac{\sigma^2(s, K_s)}{\kappa^2} < \bar{\eta}_s$, we can have $\bar{\eta}_s \geq 0$ while $\eta_s < 0$. For some stochastic models for $K$, we therefore have only stochastic profitable round trip strategies but no deterministic profitable round trip strategies.

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6 As pointed out above profitable round trip strategies exist also for $D^T_t$ being different from zero, but the relevant question in the zero spread model is the one for $D^T_T = 0$.

7 Let us also notice that, if $\eta(t, \kappa) < 0$ at some point $(t, \kappa)$ and the functions $\rho$, $\mu$ and $\sigma$ are continuous, then $\eta < 0$ in some $[t, t + \Delta t] \times [\kappa - \epsilon, \kappa + \epsilon]$. 

20
9 Concluding discussions

Summary

We propose a limit order book model with stochastic liquidity that captures random fluctuations of the limit order book depth. If the stochastic liquidity in this model follows a diffusion process meeting certain conditions, then optimal trade execution follows the classical wait region / buy region structure often observed in limit order book models with static or deterministically time dependent liquidity. For other stochastic liquidity processes, the optimal trade execution strategy can take more general forms; for example, multiple wait regions can occur, and optimal trade sizes do not need to depend monotonically on the size of the position that remains to be liquidated. In our framework, the conditions for the wait region / buy region structure also result in all round trip strategies generating positive costs even if the zero spread model is assumed.

Follow-up questions and open problems

As discussed after Theorem 8.4, the absence of profitable round trips in the zero spread model is linked to the convexity of the cost functional $J$ in the strategy. This link holds true also in the case of deterministic $K$ (at least for absolutely continuous $K$, in which case (14) applies) But in the case of deterministic $K$ we have WR-BR structure whenever $K$ is continuous (see Theorem 7.7 in Fruth, Schöneborn, and Urusov (2014)), that is, in the case of deterministic $K$ we can go beyond convexity of $J$ and still have WR-BR structure. By analogy one might expect the same in our present setting of diffusive $K$ and conjecture that the convexity assumption in Theorem 5.2 is superfluous. However, as the WR-BR counterexamples in Section reveal, this assumption cannot be dropped entirely for stochastic liquidity. This suggests follow-up questions such as:

- Is it possible to relax Assumption 5.1 and still guarantee WR-BR structure? Putting it differently, how far can one go beyond convexity of $J$ (if possible at all) and still have WR-BR structure?

- Other than (intuitive) WR-BR and (counterintuitive, but possible) WR-BR-WR structures, what kind of structures can appear in general? E.g., is it possible to have multiple buy regions?

- How is the optimal strategy changed when a different cost function $J$ is used, e.g., to incorporate risk aversion of the trader?

These questions are not easy to treat in a mathematically rigorous way and are natural open problems for future research.

A Auxiliary results

The following simple result is recalled from Fruth, Schöneborn, and Urusov (2014). It shows that our formulas for the price deviation and for the cost are economically sensible.

Lemma A.1 (Splitting argument).

Doing two separate trades $\xi_\alpha, \xi_\beta > 0$ at the same time $s$ has the same effect as trading at once $\xi := \xi_\alpha + \xi_\beta$, i.e., both alternatives incur the same cost and the same price deviation $D_{s+}$.

It is worth noting at this point that the corresponding results for deterministic $K$, Proposition 8.3 and Corollary 8.5 in Fruth, Schöneborn, and Urusov (2014), are proved differently from Theorem 8.4.
Proof. The cost is in both cases
\[
\left( D_s + \frac{K_s}{2} \xi \right) \xi = D_s(\xi_\alpha + \xi_\beta) + \frac{K_s}{2}(\xi_\alpha^2 + 2\xi_\alpha\xi_\beta + \xi_\beta^2) = \left( D_s + \frac{K_s}{2} \xi_\alpha \right) \xi_\alpha + \left( D_s + K_s \xi_\alpha + \frac{K_s}{2} \xi_\beta \right) \xi_\beta
\]
and the price deviation \( D_{n+1} = D_n(\xi_\alpha + \xi_\beta) \) after the trade is the same in both cases as well. \( \square \)

**Proposition A.2** (Continuity of the value function).
For each \( t \in [0,T] \) and \( \kappa > 0 \), the function
\[
U(t, \cdot, \kappa) : [0, \infty)^2 \to [0, \infty)
\]
is continuous.

The proof is similar to the one of Proposition 5.5 in Fruth, Schöneborn, and Urusov (2014).

**Proposition A.3** (Trading never completes early).
For all \( t \in [0,T] \), \( \delta \geq 0 \), \( x > 0 \) and \( \kappa > 0 \), the value function satisfies
\[
U(t, \delta, x, \kappa) < \left( \delta + \frac{\kappa}{2} \right) x,
\]
i.e., it is never optimal to buy the whole remaining position at any time \( t \in [0,T] \).

Proof. The result immediately follows from Lemma 3.1 and the corresponding result for deterministically varying \( K \), see Proposition 5.6 in Fruth, Schöneborn, and Urusov (2014). \( \square \)

**B Proof of the WR-BR theorem**

As discussed in Section 3, Theorem 5.2 follows from the results formulated in Subsections 5.1 and 5.2 which we now prove.

**Proof of Lemma 5.3** Let \( t, \delta \) and \( \kappa \) be fixed. Clearly, Assumption 5.1 ii) implies \( \mathbb{E}_{t, \kappa, \sup_{s \in [t,T]} K_s < \infty} \), hence \( J(\cdot) \) is finite on the whole \( \mathcal{A}_t \). We demonstrate below that
\[
J(\Theta) = \frac{1}{2} \mathbb{E}_{t, \delta, \kappa} \left[ \frac{D^2_{T+}}{K_T} \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s D^2_s ds \right]
\]
with \( (\eta_s) \) as in Assumption 5.1 i). The right-hand side is strictly convex in the process \( (D_s)_{s \in [t,T]} \). Thus, for two different strategies \( \Theta', \Theta'' \in \mathcal{A}_t \) with corresponding \( D', D'' \) both starting in \( D'_t = D''_t = \delta \), we have \( D(\nu \Theta' + (1 - \nu) \Theta'') = \nu D' + (1 - \nu) D'' \), hence \( J(\nu \Theta' + (1 - \nu) \Theta'') < \nu J(\Theta') + (1 - \nu) J(\Theta'') \) for all \( \nu \in (0,1) \) as desired. Hence, we only need to show (23).

Define the local martingale \( M_s := \int_{[t,s]} \frac{D^2_s \sigma(u, K_s)}{2K^2_s} dW^K_u \) for \( s \in [t, \infty) \). That is, \( \tau_n = \{ s \geq t \mid \langle M \rangle_s \geq n \} \) is an increasing sequence of stopping times such that \( \tau_n \nearrow \infty \) a.s. and the stopped process \( M^{\tau_n} \) is a martingale for every \( n \). In particular, \( \mathbb{E}_{t, \delta, \kappa} [M_{T^{\tau_n}}] = 0 \). Due to the monotone convergence theorem and \( \tau_n \geq T \) a.s. for large \( n \),
\[
J(\Theta) = \lim_{n \to \infty} \mathbb{E}_{t, \delta, \kappa} \left[ \int_{[t,T^{\tau_n}]} \left( D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s \right]. \quad (24)
\]
Using \( d\Theta_s = \frac{dD_s + \rho_s D_s ds}{K_s} \) and \( \Delta \Theta_s = \frac{\Delta D_s}{K_s} \), we get
\[
J(\Theta) = \lim_{n \to \infty} E_{t,\delta,\kappa} \left[ \int_{[t,T \wedge \tau_n]} \frac{D_s}{K_s} \frac{\Delta D_s}{K_s} dD_s + \int_{[t,T \wedge \tau_n]} \frac{\rho_s D_s^2}{K_s} ds + \int_{[t,T \wedge \tau_n]} \frac{1}{2} \Delta D_s \rho_s D_s ds \right].
\]
The last integral is zero because \( D \) has at most countably many jumps. With integration by parts for càglàd processes,
\[
\int_{[t,T \wedge \tau_n]} \frac{D_s}{K_s} dD_s = \frac{D_s^2}{K_s(T \wedge \tau_n)} - \delta^2 - \int_{[t,T \wedge \tau_n]} D_s d\left( \frac{D}{K} \right)_s - \sum_{s \in [t,T \wedge \tau_n]} (\Delta D_s)^2.
\]

Use \( d\left( \frac{D}{K} \right)_s = \frac{1}{K} dD_s + D_s d\left( \frac{1}{K} \right)_s \) and rearrange terms to get
\[
\int_{[t,T \wedge \tau_n]} \frac{D_s}{K_s} dD_s = \frac{1}{2} \left( \frac{D_s^2}{K_s(T \wedge \tau_n)} - \delta^2 - \int_{[t,T \wedge \tau_n]} D_s d\left( \frac{1}{K} \right)_s - \sum_{s \in [t,T \wedge \tau_n]} (\Delta D_s)^2 \right).
\]

Applying Itô’s formula
\[
d\left( \frac{1}{K} \right)_s = \left( \frac{\sigma^2(s,K_s)}{K_s^3} - \frac{\mu(s,K_s)}{K_s^2} \right) ds - \frac{\sigma(s,K_s)}{K_s^2} dW_s,
\]

yields
\[
\int_{[t,T \wedge \tau_n]} \left( D_s + \frac{K_s}{2} \Delta \Theta_s \right) d\Theta_s = \frac{1}{2} \left[ \frac{D_s^2}{K_s(T \wedge \tau_n)} - \frac{\delta^2}{\kappa} + \int_{[t,T \wedge \tau_n]} \eta_s D_s^2 ds + M_{T \wedge \tau_n} \right].
\]

Lebesgue’s dominated convergence theorem together with Assumption 5.1(ii) guarantee
\[
E_{t,\delta,\kappa} \left[ \frac{D_s^2}{K_s(T \wedge \tau_n)} \right] \quad \overset{n \to \infty}{\longrightarrow} \quad E_{t,\delta,\kappa} \left[ \frac{D_s^2}{K_T} \right],
\]
while, by the monotone convergence theorem, we have
\[
E_{t,\delta,\kappa} \left[ \int_{[t,T \wedge \tau_n]} \eta_s D_s^2 ds \right] \quad \overset{n \to \infty}{\longrightarrow} \quad E_{t,\delta,\kappa} \left[ \int_{[t,T]} \eta_s D_s^2 ds \right].
\]

Now (24) and (25) together with (26)–(27) complete the proof. \(\square\)

**Proof of Proposition 7.4** Thanks to Lemma 3.5, we only need to prove existence. Let \( t, \delta \) and \( \kappa \) be fixed. We start by showing that there exists a sequence of strategies \( (\overline{\Theta}^n) \subset \mathcal{A}_t(x) \) that converges in some sense to a strategy \( \Theta^* \in \mathcal{A}_t(x) \) and minimizes the cost \( J \), i.e., \( \lim_{n \to \infty} J(\overline{\Theta}^n) = \inf_{\Theta \in \mathcal{A}_t(x)} J(\Theta) \). We conclude by deducing that \( \lim_{n \to \infty} J(\overline{\Theta}^n) = J(\Theta^*) \).

Let \( (\Theta^j) \subset \mathcal{A}_t(x) \) be a minimizing sequence for \( J \). Due to the Komlós theorem in the form of Lemma 3.5 from Kabanov (1999), there exists a Cesaro convergent subsequence \( (\Theta^{jm}) \). That is,
\[
\overline{\Theta}^n := \frac{1}{n} \sum_{m=1}^{n} \Theta^{jm}
\]

converges to some strategy \( \Theta^* \in \mathcal{A}_t \) in the following sense. For \( \mathbb{P}_{t,\kappa} \)-almost every \( \omega \), the measures \( \overline{\Theta}^n(\omega) \) on \([t,T]\) converge weakly to the measure \( \Theta^*(\omega) \). In what follows we call such a convergence
pathwise weak convergence in time. Equivalently, for almost every \( \omega \), we have \( \lim_{n \to \infty} \Theta^n_s = \Theta^*_s \) whenever \( s \in [t, T] \) with \( \Delta \Theta^*_s = 0 \). We set \( \Theta^*_{t+} = x \) redefining \( \Theta^*_{t+} \) if necessary. Notice that this does not disturb the weak convergence. Thus, \( \Theta^* \in \mathcal{A}_t(x) \). Moreover, \( (\Theta^n_s) \subset \mathcal{A}_t(x) \) is again a minimizing sequence for \( J \) because \( J \) is convex.

It remains to show that \( \Theta^* \) attains the infimum. Applying (23) yields

\[
J(\Theta^*) = \frac{1}{2} \mathbb{E}_{t,\delta,x} \left[ \frac{(D^n_T)^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s (D^n_s)^2 \, ds \right], \tag{28}
\]

\[
J(\Theta^*) = \frac{1}{2} \mathbb{E}_{t,\delta,x} \left[ \frac{(D^*_T)^2}{K_T} - \frac{\delta^2}{\kappa} + \int_{[t,T]} \eta_s (D^*_s)^2 \, ds \right], \tag{29}
\]

where \( D^n \) and \( D^* \) are the price deviation processes that correspond to \( \Theta^n \) and \( \Theta^* \). By the (pathwise weak in time) convergence of \( \Theta^n \) to \( \Theta^* \), for almost every \( \omega \), we get \( \lim_{n \to \infty} D^n_s = D^*_s \) for every point \( s \in [t, T] \), where \( \Theta^* \) is continuous, as well as for \( s = T+ \). Fatou’s lemma and (28)–(29) now imply \( J(\Theta^*) \leq \lim \inf_{n \to \infty} J(\Theta^n) \), which means that \( \Theta^* \) is an optimal strategy. \( \square \)

**Proposition B.1.** (WR-BR structure is equivalent to trading towards the barrier).

Assume that for each \( (t,\delta,x,\kappa) \) there exists a unique optimal strategy

\[
(\Theta^*_x(t,\delta,x,\kappa))_{s \in [t,T+]} \in \mathcal{A}_t(x).
\]

Then the following statements are equivalent.

(a) The value function has WR-BR structure.

(b) There exists \( \tilde{c} : [0, T) \times (0, \infty) \to (0, \infty) \) such that for all \( (t,\delta,x,\kappa) \)

\[
\Delta \Theta^*_x(t,\delta,x,\kappa) = \max \left\{ 0, \frac{x - \tilde{c}(t,\kappa)\delta}{1 + \kappa \tilde{c}(t,\kappa)} \right\}.
\]

In particular, \( \Delta \Theta^*_x(t,\delta,x,\kappa) \) is continuous in \( \delta \) and \( x \).

(c) For all \( (t,\delta,\kappa) \), the function \( x \mapsto \Delta \Theta^*_x(t,\delta,x,\kappa) \) is increasing.

Furthermore, if these equivalent statements hold, then the function \( \tilde{c} \) in (b) coincides with the barrier \( c \) of Definition 4.3.

**Proof.** First we prove the equivalence of (a) and (b). Statement (c) follows immediately from (b). We conclude by showing that (c) implies (b). The scaling property (Lemma 4.1) yields

\[
\Delta \Theta^*_x(t,\delta,x,\kappa) = \delta \Delta \Theta^*_x \left( t, 1, \frac{x}{\delta}, \kappa \right).
\]

Therefore, we only need to discuss the case \( \delta = 1 \). Fix arbitrary \( t \in [0,T], \kappa \in (0,\infty) \).

(a) \( \Rightarrow \) (b) The assertion holds for \( x = 0 \). Assume \( x \in [0,c(t,\kappa)] \), where \( c \) is the barrier in Definition 4.3. Then the WR-BR structure implies that for all \( \xi \in (0,x) \)

\[
U(t,1,x,\kappa) < U(t,1,\kappa \xi, x - \xi, \kappa) + \left( 1 + \frac{\kappa}{2} \xi \right) \xi.
\]

Therefore it cannot be optimal to trade immediately at time \( t \).

\( ^9 \)See also Lemma 7.3 of Fruth, Schöneborn, and Urusov (2014).
Assume $c(t, \kappa) < \infty$ and $x \in (c(t, \kappa), \infty)$. Then the WR-BR structure implies that there exists $\xi \in (0, x)$ such that
\[
U(t, 1, x, \kappa) = U\left(t, 1 + \kappa \xi, x - \xi, \kappa\right) + \left(1 + \frac{\kappa}{2} \xi\right) \xi.
\]
Due to the uniqueness of the optimal strategy, we get
\[
\Delta \Theta^*_t (t, 1, x, \kappa) = \xi + \frac{\kappa}{1 + \kappa \xi} \xi > 0.
\]
For $0 < \xi < \frac{x-c(t, \kappa)}{1 + \kappa c(t, \kappa)}$, we have $\frac{x-c(t, \kappa)}{1 + \kappa c(t, \kappa)} > c(t, \kappa)$ and thus
\[
\Delta \Theta^*_t \left(t, 1 + \kappa \xi, x - \xi, \kappa\right) > 0.
\]
Consequently, $\Delta \Theta^*_t (t, 1, x, \kappa) \geq \frac{x-c(t, \kappa)}{1 + \kappa c(t, \kappa)}$. Two trades executed immediately after each other have the same effect as one trade of their combined size (see Lemma A.1). Due to this splitting argument, we have
\[
\Delta \Theta^*_t (t, 1, x, \kappa) = \frac{x-c(t, \kappa)}{1 + \kappa c(t, \kappa)} + \Delta \Theta^*_t \left(t, 1 + \kappa \xi, x - x - c(t, \kappa)\right).
\]
Observe that the second summand equals zero because
\[
\frac{x - \frac{x-c(t, \kappa)}{1 + \kappa c(t, \kappa)}}{1 + \kappa c(t, \kappa)} = c(t, \kappa).
\]
Thus, we proved (b) with $c(t, \kappa) = c(t, \kappa)$.

(b) $\Rightarrow$ (a) Assume $x \in (0, \tilde{c}(t, \kappa)]$. Then (30) implies $\Delta \Theta^*_t (t, 1, x, \kappa) = 0$. Together with the uniqueness of the optimal strategy we can therefore conclude that $x \notin B_{r_{t, \kappa}}$, as, for all $\xi \in (0, x)$,
\[
U(t, 1, x, \kappa) < U\left(t, 1 + \kappa \xi, x - \xi, \kappa\right) + \left(1 + \frac{\kappa}{2} \xi\right) \xi.
\]
Assume $\tilde{c}(t, \kappa) < \infty$ and $x \in (\tilde{c}(t, \kappa), \infty)$. Then (30) implies
\[
\Delta \Theta^*_t (t, 1, x, \kappa) \in (0, x).
\]
The optimality of $\Theta^*$ leads to the conclusion $x \in B_{r_{t, \kappa}}$ because
\[
U(t, 1, x, \kappa) = U\left(t, 1 + \kappa \Delta \Theta^*_t (t, 1, x, \kappa), x - \Delta \Theta^*_t (t, 1, x, \kappa), \kappa\right)
\]
\[
+ \left(1 + \frac{\kappa}{2} \Delta \Theta^*_t (t, 1, x, \kappa)\right) \Delta \Theta^*_t (t, 1, x, \kappa).
\]
We thus established that $B_{r_{t, \kappa}} = (\tilde{c}(t, \kappa), \infty)$, i.e., we have WR-BR structure with the barrier $c(t, \kappa) = \tilde{c}(t, \kappa)$.

(c) $\Rightarrow$ (b) Define
\[
\tilde{c}(t, \kappa) := \inf \{x \in (0, \infty) | \Delta \Theta^*_t (t, 1, x, \kappa) > 0\},
\]
where $\inf \emptyset := \infty$. We are done for $\tilde{c}(t, \kappa) = \infty$. Let $\tilde{c}(t, \kappa) < \infty$. Then the definition of $\tilde{c}(t, \kappa)$ guarantees $\Delta \Theta^*_t (t, 1, x, \kappa) = 0$ for all $x < \tilde{c}(t, \kappa)$, and Property (c) implies $\Delta \Theta^*_t (t, 1, x, \kappa) > 0$ for all $x > \tilde{c}(t, \kappa)$. Suppose for a contradiction that
\[
\Delta \Theta^*_t (t, 1, \tilde{c}(t, \kappa), \kappa) > 0.
\]
Due to the uniqueness and the splitting argument, we then have, for $\epsilon \in (0, \Delta \Theta^*_t (t, 1, \tilde{c}(t, \kappa), \kappa))$,
\[
\Delta \Theta^*_t (t, 1, \tilde{c}(t, \kappa), \kappa) = \epsilon + \Delta \Theta^*_t (t, 1 + \kappa \epsilon, \tilde{c}(t, \kappa) - \epsilon, \kappa) = \epsilon < \Delta \Theta^*_t (t, 1, \tilde{c}(t, \kappa), \kappa).
\]
Therefore, \( \Delta \Theta^*_t(t, 1, x, \kappa) = 0 \) for all \( x \leq \bar{c}(t, \kappa) \).

We still need to prove \( \Delta \Theta^*_t(t, 1, x, \kappa) = \frac{x - \bar{c}(t, \kappa)}{1 + \kappa \bar{c}(t, \kappa)} \) for \( x > \bar{c}(t, \kappa) \). Let us first assume that \( \Delta \Theta^*_t(t, 1, x, \kappa) > \frac{x - \bar{c}(t, \kappa)}{1 + \kappa \bar{c}(t, \kappa)} \). Once more, we make use of the uniqueness and the splitting argument in order to get a contradiction

\[
\Delta \Theta^*_t(t, 1, x, \kappa) = \frac{x - \bar{c}(t, \kappa)}{1 + \kappa \bar{c}(t, \kappa)} + \Delta \Theta^*_t \left( t, 1 + \kappa \bar{c}(t, \kappa), x - \frac{x - \bar{c}(t, \kappa)}{1 + \kappa \bar{c}(t, \kappa)} \right) < \Delta \Theta^*_t(t, 1, x, \kappa). 
\]

Finally, assume \( \Delta \Theta^*_t(t, 1, x, \kappa) < \frac{x - \bar{c}(t, \kappa)}{1 + \kappa \bar{c}(t, \kappa)} \).

That is, \( \frac{x - \Delta \Theta^*_t(t, 1, x, \kappa)}{1 + \kappa \bar{c}(t, \kappa)} > \bar{c}(t, \kappa) \) and we again arrive at a contradiction:

\[
\Delta \Theta^*_t(t, 1, x, \kappa) = \Delta \Theta^*_t(t, 1, x, \kappa) + \Delta \Theta^*_t \left( t, 1 + \kappa \Delta \Theta^*_t(t, 1, x, \kappa), x - \Delta \Theta^*_t(t, 1, x, \kappa), \kappa \right) > \Delta \Theta^*_t(t, 1, x, \kappa).
\]

This concludes the proof.

**Proof of Proposition 5.4** According to Propositions 5.4 and 5.1, we only need to show that the optimal initial trade \( \Delta \Theta^*_t(t_n, \delta, x, \kappa) \) is increasing in \( x \), where \( \Theta^* \) denotes the corresponding optimal strategy. Due to the scaling property of the value function (Lemma 4.1),

\[
\Delta \Theta^*_t(t_n, \delta, x, \kappa) = \delta \Delta \Theta^*_t \left( t_n, 1, \frac{x}{\delta}, \kappa \right).
\]

Due to the splitting argument (Lemma A.1) and the uniqueness of the optimal strategy, \( \Delta \Theta^*_t(t_n, 1, \cdot, \kappa) \) must be increasing and continuous apart from a possible discontinuity in the form of a jump back to zero. That is, there might exist \( y > 0 \) with \( \Delta \Theta^*_t(t_n, 1, y-, \kappa) > 0 \) and \( \Delta \Theta^*_t(t_n, 1, y+, \kappa) = 0 \). In the following, we exclude such discontinuities using a Komlós argument as in the proof of Proposition 5.4.

Suppose for a contradiction that such a discontinuity exists in \( y > 0 \). Let us take some monotone sequences \( y^{1,j} \searrow y \) and \( y^{2,j} \nearrow y \) and define \( \Theta^{i,j} := \Theta^*(t_n, 1, y^{i,j}, \kappa) \) for \( i \in \{1, 2\} \). Let us choose \( \epsilon > 0 \) such that \( \Delta \Theta^{i,j}_n \geq \epsilon > 0 \) for all sufficiently large \( j \). Without loss of generality we assume that the latter inequality holds for all \( j \). As \( V^{\text{dis}} \) is continuous in \( y \) (see Proposition A.2),

\[
J \left( t_n, 1, \Theta^{1,j}, \kappa \right) = V^{\text{dis}} \left( t_n, y^{1,j}, \kappa \right) \xrightarrow{j \to \infty} V^{\text{dis}} \left( t_n, y, \kappa \right).
\]

Define \( b_j := \frac{y}{y^{1,j}} \searrow 1 \). Then we have

\[
0 \leq J \left( t_n, 1, b_j \Theta^{1,j}, \kappa \right) - J \left( t_n, 1, \Theta^{1,j}, \kappa \right) \\
\leq J \left( t_n, b_j, b_j \Theta^{1,j}, \kappa \right) - J \left( t_n, 1, \Theta^{1,j}, \kappa \right) = (b_j^2 - 1) J \left( t_n, 1, \Theta^{1,j}, \kappa \right) \xrightarrow{j \to \infty} 0.
\]

Therefore, \( \left( b_j \Theta^{1,j} \right) \) is a minimizing sequence of strategies that build up the position of \( y \) shares, i.e., \( b_j \Theta^{1,j} \in A^{\text{dis}}_t(y) \) and

\[
\lim_{j \to \infty} J \left( t_n, 1, b_j \Theta^{1,j}, \kappa \right) = V^{\text{dis}} \left( t_n, y, \kappa \right).
\]

As in the proof of Proposition 5.4, we can define \( \overline{\Theta} \in A^{\text{dis}}_t(y) \) as the pathwise weak in time limit of the averaged sum over a subsequence of \( \left( b_j \Theta^{1,j} \right) \) such that \( J(t_n, 1, \overline{\Theta}, \kappa) = V^{\text{dis}}(t_n, y, \kappa) \), i.e., \( \overline{\Theta} \) is an optimal strategy. Due to the construction of \( \overline{\Theta} \), with \( \epsilon > 0 \) from above, we have

\[
\Delta \overline{\Theta}_t(t_n, 1, y, \kappa) \geq \epsilon > 0.
\]
Similarly, one constructs an optimal strategy $\mathbf{\Theta} \in \mathcal{A}_{ts}^{dis}(y)$ using the sequence $\left(\frac{\kappa}{\kappa \text{th}} \Theta_{n}^{\geq j}\right)$ of strategies with zero initial trade: as we now treat the discrete time case, the initial trade remains zero also in the weak limit
\[
\Delta \mathbf{\Theta}_{ts}(t_{n}, 1, y, \kappa) = 0.
\]
Thus, $\mathbf{\Theta}$ and $\mathbf{\Theta}$ are different. This contradicts the uniqueness of the optimal strategy. \hfill \Box

**Lemma B.2.** (Approximation via step functions).

Let Assumption 5.1 ii)–iii) hold. For $\Theta \in \mathcal{A}_{cts}^{dis}(x)$, let $\Theta^{N} \in \mathcal{A}_{cts}^{dis}(x)$ be its approximation from below by an equidistant grid step function. More precisely, define $T_{N}^{(1)} := \{t, T\}$,
\[
T_{N}^{(1)} := \{s + \frac{T - t}{2^{N}} \}\wedge T \quad \text{if} \quad s \in \mathcal{T}_{N}^{(1)},
\]
and
\[
\Theta_{s}^{N} := \begin{cases} 
\Theta_{u}^{+} & \text{if } s = t \\
\Theta_{x} & \text{if } s \in (u, u + \frac{T - t}{2^{N}}], u \in \mathcal{T}_{N}^{(1)} 
\end{cases}.
\]

Then $J(t, 1, \Theta, \kappa) = \lim_{N \to \infty} J(t, 1, \Theta^{N}, \kappa)$.

**Proof.** We proceed as in the end of the proof of Proposition 5.4. That is, we only need to show that $\Theta^{N}$ converges pathwise weakly in time to $\Theta$. Due to $\mathcal{T}_{N}^{(1)} \subset \mathcal{T}_{N}^{(1)}, \Theta^{N}$ is increasing in $N$. For all $s \in [t, T]$, the sequence $(\Theta_{s}^{N})_{N \in \mathbb{N}}$ is bounded above by $\Theta_{s}$. Hence, it is convergent. Due to the definition of $\Theta^{N}$, we must even have $\lim_{N \to \infty} \Theta_{s}^{N} = \Theta_{s}$ for all $s \in [t, T]$ with $\Delta \Theta_{s} = 0$. Now the result follows from (23) and the dominated convergence theorem (apply Assumption 5.1 ii) and iii)). \hfill \Box

**Lemma B.3.** (Cesaro weak convergence).

Fix $t \in [0, T], \kappa \in (0, \infty)$ and for various $x \in [0, \infty)$ consider
\[
(\Theta^{N}(t, 1, x, \kappa))_{N \in \mathbb{N}} \subset \mathcal{A}_{cts}^{dis}(x).
\]

Then there exists a subsequence $N_{j}(t, x, \kappa)$, which does not depend on $x$, and a set of strategies $\hat{\Theta}(t, 1, \cdot, \kappa)$ such that for all $x \in [0, \infty) \cap \mathbb{Q}$
\[
\frac{1}{m} \sum_{j=1}^{m} \Theta_{N_{j}}^{(1)}(t, 1, x, \kappa) \xrightarrow{w} \hat{\Theta}(t, 1, x, \kappa).
\]

In (31) the notation $\xrightarrow{w}$ stands for the pathwise weak convergence in time (cf. the proof of Proposition 5.4).

**Proof.** As $\mathbb{Q}$ is countable, we can write $[0, \infty) \cap \mathbb{Q} = \{x_{1}, x_{2}, \ldots\}$. For each $x \in [0, \infty)$, the Komlós theorem guarantees the existence of a subsequence $N_{j}(t, x, \kappa)$ such that the desired pathwise weak convergence in time holds. That is we get $(N_{j})_{j \in \mathbb{N}} \subset \mathbb{N}$ for $x_{1}$ and extract the subsequence $N_{j}^{(2)}$ for $x_{2}$ from $N_{j}^{(1)}$, etc. We remark that the Komlós theorem gives not only Cesaro convergent subsequences, but subsequences such that all their subsequences are Cesaro convergent to the same limit. The Cantor diagonal sequence $N_{j} := N_{j}^{(1)}$ then guarantees the Cesaro weak convergence of $\Theta_{N_{j}}^{(1)}(t, 1, x, \kappa)$ for all $x \in [0, \infty) \cap \mathbb{Q}$.

**Proof of Proposition 5.5.** As in the proof of Proposition 5.5 we only need to exclude the jump back to zero of $x \mapsto \Delta \Theta_{ts}(t, 1, x, \kappa)$. Let $\Theta_{N}^{*} \in \mathcal{A}_{cts}^{dis}(x)$ be the approximation of $\Theta^{*} \in \mathcal{A}_{cts}^{dis}(x)$ by step functions from below as in Lemma B.2 Then
\[
J(t, 1, \Theta^{*}, \kappa) = \lim_{N \to \infty} J(t, 1, \Theta^{N}, \kappa).
\]

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Let $\Theta^{*N}$ be the unique optimal strategy within $\mathcal{A}^{\text{dis}}(x)$ for the time grid $T^N$, i.e.,

$$J(t, 1, \Theta^N, \kappa) \geq J(t, 1, \Theta^{*N}, \kappa) \geq J(t, 1, \Theta^*, \kappa).$$

Hence,

$$J(t, 1, \Theta^*, \kappa) = \lim_{N \to \infty} J(t, 1, \Theta^{*N}, \kappa).$$

That is, for each $x \in [0, \infty)$, $(\Theta^{*N}(t, 1, x, \kappa))_{N \in \mathbb{N}}$ is a minimizing sequence, and for each $N \in \mathbb{N}$, $x \mapsto \Delta \Theta^{*N}_t(t, 1, x, \kappa)$ is increasing thanks to Proposition 5.3.

Apply Lemma B.3 to $\Theta^{*N}(t, 1, x, \kappa)$ (for all rational $x$). The resulting strategy $\tilde{\Theta}(t, 1, x, \kappa)$ as in (31) is optimal (apply convexity of the cost function together with (23) and the dominated convergence theorem). As the optimal strategy is unique, $\tilde{\Theta}(t, 1, x, \kappa)$ must coincide with $\Theta^*(t, 1, x, \kappa)$ for all $x \in [0, \infty) \cap \mathbb{Q}$. Furthermore, as we already proved WR-BR structure in discrete time, for all $N$ and $s \in [t, T]$, the function $x \mapsto \Theta^*_N(t, 1, x, \kappa)$ is increasing. Due to the pathwise weak convergence as in (31), for all $s \in [t, T]$, the function $x \mapsto \Theta^*_s(t, 1, x, \kappa)$ is increasing over rational $x$. In particular, $x \mapsto \Delta \Theta^*_s(t, 1, x, \kappa) \equiv \Theta^*_s(t, 1, x, \kappa)$ is increasing over rational $x$. As we only need to exclude the downward jump, it suffices to have this monotonicity over the rational numbers.

C Proofs for GBM and CIR cases with WR-BR structure

Proof of Proposition 6.2

i) We have $\eta_t = \frac{1}{\kappa_t} (2\bar{\mu} + \bar{\mu} - \bar{\sigma}^2) > 0$.

ii) Set $q_t := \frac{1}{\kappa_t}$. Thanks to Hölder’s inequality,

$$\mathbb{E}_{0,\kappa} \left[ \left( \sup_{t \in [0, T]} K_t \right)^2 \right] \leq \mathbb{E}_{0,\kappa} \left[ \sup_{t \in [0, T]} K_t^4 \right] \frac{1}{\mathbb{E}_{0,\kappa} \left[ \hat{W}_t \right]}. \quad (32)$$

The explicit formula for GBM, $K_t = K_0 e^{\bar{\sigma} \hat{W}_t + (\bar{\mu} - \frac{\bar{\sigma}^2}{2}) t}$, yields

$$\mathbb{E}_{0,\kappa} \left[ \sup_{t \in [0, T]} K_t^4 \right] \leq \kappa^4 \max \left\{ 1, e^{4(\bar{\mu} - \frac{\bar{\sigma}^2}{2}) T} \right\} \mathbb{E}_{0,\kappa} \left[ \exp \left( 4\bar{\sigma} \sup_{t \in [0, T]} \hat{W}_t \right) \right].$$

The latter expression is finite due to the fact that $(\sup_{t \in [0, T]} \hat{W}_t)$ has the same distribution as $|\hat{W}_T|$, which is a consequence of the reflection principle for a Brownian motion. The second expectation in (32) is finite because $q_t = \frac{1}{\kappa_t}$ is also a GBM (with drift $(\bar{\sigma}^2 - \bar{\mu})$ and volatility $\bar{\sigma}$).

iii) Due to the form of $\eta_t$, it is enough to consider

$$\mathbb{E}_{0,\kappa} \left[ \int_0^T \left( \sup_{t \in [0, T]} K_t \right)^2 \frac{1}{K_t} dt \right] \leq T \mathbb{E}_{0,\kappa} \left[ \left( \sup_{t \in [0, T]} K_t \right)^2 \right] \left( \frac{1}{\mathbb{E}_{0,\kappa} \left[ \inf_{t \in [0, T]} K_t \right]} \right),$$

where the right-hand side is finite according to ii).

Proof of Proposition 6.3

Such a CIR process stays a.s. strictly positive, as the Feller condition $\bar{\mu} \bar{K} \geq \bar{\sigma}^2/2$ is met. Moreover, it turns out that $\eta_t = \frac{1}{\kappa_t} (2\bar{\mu} - \bar{\mu}) + \frac{1}{\kappa_t} (\bar{\mu} \bar{K} - \bar{\sigma}^2) > 0$ due to our assumptions. Conditions ii) and iii) both hold by showing

$$\mathbb{E}_{0,\kappa} \left[ \left( \sup_{t \in [0, T]} K_t \right)^2 \right] < \infty.$$
Thanks to Hölder’s inequality, with $q_t = \frac{1}{K_t}$, we have

$$E_{0, \kappa} \left[ \frac{\left( \sup_{t \in [0,T]} K_t \right)^4}{\left( \inf_{t \in [0,T]} K_t \right)^4} \right] \leq E_{0, \kappa} \left[ \sup_{t \in [0,T]} K_t^4 \right] \frac{4}{\sup_{t \in [0,T]} \left( \inf_{t \in [0,T]} K_t \right)^4} .$$

(33)

As the drift of the CIR process is bounded above, we can isolate the local martingale part of $K$ and use the Burkholder-Davis-Gundy inequalities. With appropriate positive constants $c_n$, we obtain

$$E_{0, \kappa} \left[ \sup_{t \in [0,T]} K_t^4 \right] \leq c_1 \left\{ \kappa^8 + (\bar{\mu}K)^8 + E_{0, \kappa} \left[ \sup_{t \in [0,T]} \left( \int_0^t \sigma^2 K_s dW^K_s \right)^8 \right] \right\}$$

$$\leq c_2 \left\{ \kappa^8 + (\bar{\mu}K)^8 + E_{0, \kappa} \left( \int_0^T \sigma^2 K_s ds \right)^4 \right\} .$$

(34)

The latter expectation is finite because all positive moments of the CIR process are finite (e.g., see Filipovic and Mayerhofer (2009)).

It remains to show that the second term on the right-hand side of (33) is finite. By Itô’s formula, the process $q_t = \frac{1}{K_t}$ has the dynamics

$$dq_t = (\bar{\mu} q_t - (\bar{\mu} K - \bar{\alpha}^2) q_t^3) \, dt - \bar{\alpha}^3 q_t^2 \, dW^K_t .$$

With these preparations, we proceed similarly to (34):

$$E_{0, \kappa} \left[ \sup_{t \in [0,T]} q_t^4 \right] \leq c_3 \left\{ \kappa^{-4} + \left( \bar{\mu}^2 T \right) \frac{4}{(\bar{\mu} K - \bar{\alpha}^2)} \right\} + E_{0, \kappa} \left[ \sup_{t \in [0,T]} \left( \int_0^t \bar{\sigma}^2 q_s^2 \, dW^K_s \right)^4 \right] \right\}$$

$$\leq c_4 \left\{ \kappa^{-4} + \left( \bar{\mu}^2 T \right) \frac{4}{(\bar{\mu} K - \bar{\alpha}^2)} \right\} + E_{0, \kappa} \left( \int_0^T \bar{\sigma}^2 q_s^4 \, ds \right)^4 \right\} .$$

We are done because $E_{0, \kappa} \left( \int_0^T q_s^4 \, ds \right)^4 \leq c_5 \int_0^T E_{0, \kappa} \left[ q_s^4 \right] \, ds$, and the fourth moment of the inverse CIR process is finite whenever $\bar{\mu} K > 2\bar{\alpha}^2$ (e.g., see Ahn and Gao (1999) for an explicit calculation of negative moments of the CIR process).

D Proof of Proposition 7.2

The optimal strategy is determined by $\xi_0$, $\xi^A$ and $\xi^B$. As $c(t_1, \kappa_1) = c(t_1, \kappa_1) = 1 =: c(t_1)$ by Proposition 7.1, we see that $\xi^A > 0$ and only if $\xi^B > 0$.

Let us now consider a given trade $\xi_0$ at time $t_0$ and assume optimal trading thereafter. This results in a cost of

$$\bar{U}^{dis}(t_0, \delta, x, \kappa_0; \xi_0) := \left( \delta + \frac{\kappa_0}{2} \xi_0 \right) \xi_0 + E \left[ U^{dis}(t_1, (\delta + \kappa_0 \xi_0) a_0, x - \xi_0, \kappa_1) \right] .$$

10For every $m > 0$, there exist universal positive constants $k_m$ and $K_m$ such that

$$k_m E \left[ (M^m)^m \right] \leq E \left[ \max_{1 \leq i \leq m} |M^m| \right] \leq K_m E \left[ (M^m)^m \right]$$

for every continuous local martingale $M$ with $M_0 = 0$ and every stopping time $\tau$. E.g., see Karatzas and Shreve (2000), Chapter 3, Theorem 3.28.
It is easy to see that \( \hat{U}^{\text{dis}} \) is piecewise quadratic in \( \xi_0 \). For the section of \( \xi_0 \) where the optimal \( \xi_1 \) is positive (\( \xi_1 > 0 \)), a straightforward calculation shows that the quadratic coefficient is negative. \( U^{\text{dis}} \) therefore cannot attain its minimum in the interior of this section; the optimal strategy therefore satisfies \( \xi_0 = 0 \) or \( \xi_1 = 0 \).

Using Proposition 7.1, we easily calculate that for trading only at times \( t_0 \) and \( t_2 \), we have

\[
c_1 := c^{0.2}(t_0, \kappa_0) < a_0 = c(t_1) a_0.
\]

Hence \( (c_1, a_0) \) must be a subset of the buy region \( B_{\Theta_0} \). For \( y > a_0 \), we need to compare the cost \( U^{\text{0.2}} \) of optimally trading only at times \( t_0 \) and \( t_2 \) with the cost \( U^{1.2} \) of optimally trading only at times \( t_1 \) and \( t_2 \). Using the parameter values given in Figure 2, we find that the quadratic coefficient of \( U^{0.2} \) is larger than the quadratic coefficient of \( U^{1.2} \); therefore there must be an intersection point \( c_u > c_i \) where \( U^{1.2} = U^{0.2} \). We then have for \( y \leq c_i \) that \( U^{1.2} = U^{0.2} \) and the optimal strategy trades neither at \( t_0 \) nor \( t_1 \), for \( c_i < y < c_u \) that \( U^{0.2} < U^{1.2} \) and the unique optimal strategy trades at \( t_0 \) but not at \( t_1 \), for \( y = c_u \) that \( U^{0.2} = U^{1.2} \), and there are two optimal strategies (one trading at \( t_0 \) but not \( t_1 \), and one trading at \( t_1 \) but not \( t_0 \)), and for \( y > c_u \) that \( U^{0.2} > U^{1.2} \) and the unique optimal strategy trades at \( t_1 \) but not at \( t_0 \).

**E Proof of Theorem 8.4**

We can extend the proof of (23) to the zero spread model and find that the cost function \( J^\dagger \) satisfies

\[
J^\dagger(t, \delta, \Theta^\dagger, \kappa) = \frac{1}{2} \mathbb{E}_{t, \delta, \kappa} \left[ \frac{(D^\dagger_{T^\dagger})^2}{K_T} - \frac{\delta^2}{K} + \int_{[t,T]} \eta_s (D^\dagger_s)^2 ds \right]
\]

with \( \eta_s \equiv \eta(s, K_s) \) as in Assumption 5.1 ii). More precisely, instead of monotone convergence in (24), we need to use dominated convergence, which applies because \( \Theta \) and \( \Theta^\dagger \) are bounded and \( \mathbb{E}_{t, \kappa} [\sup_{s \in [t,T]} K_s] < \infty \) (the latter follows from Assumption 5.1 iii)), and again dominated convergence works in (26) (based on Assumption 5.1 ii)). As for (27), we use monotone convergence in the first case (\( \eta \geq 0 \)), while dominated convergence applies in the second case (due to Assumption 5.1 iii)). In particular, when we start at time \( t \) with \( D^\dagger_t = 0 \), we have

\[
J^\dagger(t, 0, \Theta^\dagger, \kappa) = \frac{1}{2} \mathbb{E}_{t, 0} \left[ \frac{(D^\dagger_{T^\dagger})^2}{K_T} + \int_{[t,T]} \eta_s (D^\dagger_s)^2 ds \right],
\]

which establishes the statement in the first case (\( \eta \geq 0 \) everywhere).

Similarly to (35), we establish that, for any stopping time \( \tau \) with \( t \leq \tau \leq T \), it holds that

\[
J^\dagger(t, \delta, \Theta^\dagger, \kappa) = \frac{1}{2} \mathbb{E}_{t, \delta, \kappa} \left[ \frac{(D^\dagger_{\tau^\dagger})^2}{K_{\tau}} - \frac{\delta^2}{K} + \int_{[t,\tau]} \eta_s (D^\dagger_s)^2 ds \right] + \mathbb{E}_{t, \delta, \kappa} \left[ \int_{(\tau,T)} \left( D^\dagger_s + \frac{K_s}{2} \Delta \Theta^\dagger_s \right) d\Theta^\dagger_s \right].
\]

We now make use of (36) to construct a profitable round trip strategy in the second case. Starting at \( (t, \kappa) \) with \( D^\dagger_t = 0 \), let us define the stopping time

\[
\tau := (t + \Delta t) \wedge \inf \left\{ s \geq t \mid K_s \notin (\kappa - \epsilon, \kappa + \epsilon) \right\}
\]

and consider the following trading strategy. First, buy \( x > 0 \) units of the asset at time \( t \). This results in \( D^\dagger_{t + \Delta t} = \kappa x \). At time \( \tau \), we have \( D^\dagger_\tau = \kappa x e^{-\int_\tau^{t + \Delta t} \rho_s \, ds} \) and sell \( y = \frac{D^\dagger_\tau}{K_{\tau}} \) units of the asset, resulting in \( D^\dagger_{\tau + \Delta t} = 0 \). We do nothing in \( (\tau, t + \Delta t) \) and then liquidate the position \( x - y \) with a uniform speed between \( t + \Delta t \) and \( T \). Notice that the position \( x - y \) is random (it depends on \( K_\tau \)), but bounded.
(due to the construction of $\tau$). Summarizing, we consider the following round trip strategy: $\Theta^t_\tau = 0$, $\Theta^\tau_s = x$ for $s \in (t, \tau]$, $\Theta^s_{t+\Delta t} = x - y$ for $s \in (\tau, t+\Delta t)$,

$$\Theta^s_{t+\Delta t} = x - y + \frac{s - t - \Delta t}{T - t - \Delta t}(y - x), \quad s \in (t + \Delta t, T],$$

and $\Theta^T_{t+} = \Theta^T_t = 0$. An application of (36) in this case yields

$$J^t_{\tau}(t, 0, \Theta^t_\tau, \kappa) = \frac{1}{2} \mathbb{E}_{t,0,\kappa} \left[ \int_{[t,\tau]} \eta_s(D^s_t)^2 ds \right] + \mathbb{E}_{t,0,\kappa} \left[ \int_{(t+\Delta t, T]} D^s_{t+\Delta t} d\Theta^s_{t+\Delta t} \right].$$

The first term on the right-hand side is strictly negative (we are considering the second case) and does not depend on $T$. Below we present a calculation showing that the second term goes to zero as $T$ goes to infinity, which means that, for a sufficiently large $T$, we constructed a round trip strategy with strictly negative cost, i.e., with strictly positive profit.

Relying on Assumption (A) we finally show that

$$\mathbb{E}_{t,0,\kappa} \left[ \int_{(t+\Delta t, T]} D^s_{t+\Delta t} d\Theta^s_{t+\Delta t} \right] \xrightarrow{T \to \infty} 0$$

(37)

for the strategy described above. Recall that $D^\tau_{t+} = 0$, hence $D^s_{t+\Delta t+} = 0$. That is, for $s \in (t + \Delta t, T]$, we have

$$D^s_{t+\Delta t} = \frac{y - x}{T - t - \Delta t} \int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r dr} du,$$

therefore,

$$\mathbb{E}_{t,0,\kappa} \left[ \int_{(t+\Delta t, T]} D^s_{t+\Delta t} d\Theta^s_{t+\Delta t} \right] = \mathbb{E}_{t,\kappa} \left[ \frac{(y - x)^2}{(T - t - \Delta t)^2} \int_{t+\Delta t}^T \int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r dr} du ds \right] \leq \frac{\text{const}}{(T - t - \Delta t)^2} \mathbb{E}_{t,\kappa} \left[ \int_{t+\Delta t}^T \int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r dr} du ds \right],$$

(38)

where we used that the random variable $(y - x)^2$ is bounded. Further,

$$\mathbb{E}_{t,\kappa} \left[ \int_{t+\Delta t}^T \int_{t+\Delta t}^s K_u e^{-\int_u^s \rho_r dr} du ds \right] = \int_{t+\Delta t}^T \left( \int_{t+\Delta t}^s e^{-\int_u^s \rho_r dr} du \right) \mathbb{E}_{t,\kappa}[K_u] du \leq \frac{1}{\rho} \int_{t+\Delta t}^T \mathbb{E}_{t,\kappa}[K_u] du \leq \text{const } (T - t - \Delta t).$$

Together with (38), we obtain (37). This completes the proof.

References


