Optimal Stopping via Measure Transformation: The Beibel-Lerche Approach

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Abstract

Optimal stopping of diffusions and related processes is usually done by solving a free boundary problem. In this paper, we propagate an alternative way, which has already been described in two earlier papers of Beibel and Lerche [5, 6]; we call it the B-L approach. It can be viewed as optimal stopping via measure transformation. While we emphasized in [5] a rather algebraic view, we describe here more the analytic side of the approach. Finally, it is related to some recent Jamshidian’s results on a duality in optimal stopping.

Key words and phrases: optimal stopping, the B-L approach, repeated significance test, disruption problem, Russian option, lookback option, integral option, Jamshidian’s multiplicative minimax duality.

1 Introduction

The free boundary approach is the common method to solve optimal stopping problems for stochastic processes in continuous time and with infinite horizon. One derives differential equations for the value functions in the continuation region under continuous and smooth fit conditions for the unknown boundary. Explicit solutions can most often be obtained when the resulting equations are ODEs (rather than PDEs). This is usually the case when the initial problem can be reduced to an optimal stopping problem which is stationary in time.

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The approach considered in this paper (we call it the Beibel–Lerche approach) is an alternative way to solve optimal stopping problems for processes with continuous time and infinite horizon. Originally it was developed for finding optimal or asymptotically optimal stopping times arising in sequential statistics. For instance, it works in sequential change point problems with unknown drift, when one has to deal with a nonmarkovian structure (see, for example, [3] and [7]). Later it turned out that the B-L approach can be extended. It also works in several cases related to finance and other optimal stopping problems (see [5] and [6]).

An aspect which is not treated here is that the B-L approach works also in discrete time. Then of course overshoot problems enter additionally. An example, which is simple to formulate but nontrivial to answer, is given in [26].

What is the basic idea of the B-L approach? Let \((Z_t, \mathcal{F}_t; t \geq 0)\) denote an adapted continuous stochastic process on a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\). For simplicity we assume that \(Z_1 = \lim_{t \to 1} Z_t = 0\). One wants to find a stopping time \(T^*\) with \(E_P Z_{T^*} = \max_T E_P Z_T\), where the maximum is taken over all stopping times \(T\).

The idea is to find a continuous process \((X_t, \mathcal{F}_t; t \geq 0)\), a probability measure \(Q \ll P\) and a function \(g\) with a global maximum attained at some \(x^* \in \mathbb{R}\) such that \(g(x^*) > 0\) and

\[Z_t = g(X_t) \frac{dQ}{dP} \mid_{\mathcal{F}_t}.
\]

When one has this situation, then one has the following estimate:

\[E_P Z_T = E_P (Z_T 1_{\{T < \infty\}}) = E_P \left( g(X_T) \frac{dQ}{dP} \mid_{\mathcal{F}_T} 1_{\{T < \infty\}} \right) = E_Q \left( g(X_T) 1_{\{T < \infty\}} \right) \leq g(x^*) Q(T < \infty) \leq g(x^*).
\]

With \(T^* = \inf \{t \geq 0 \mid X_t = x^*\}\) (as usual \(\inf \emptyset = \infty\)) the inequalities become equalities whenever \(Q(T^* < \infty) = 1\), and the problem is solved.

Since the Radon–Nikodym derivative \(M_t = \frac{dQ}{dP} \mid_{\mathcal{F}_t}\) is a nonnegative martingale with \(M_0 = 1\) and vice versa (for simplicity assume that the \(\sigma\)-field
\( \mathcal{F}_0 \) is trivial), it is equivalent to look for an appropriate martingale \( M \) when one searches for a measure \( Q \).

Of course this idea can also be extended to two-sided situations (see [5]). The B-L approach can be applied to many stopping problems. Beside those discussed in this paper, [5], and [6], we mention the cusum test [2], and the disruption with exponential penalty [4] as technically outstanding examples.

This paper is mainly tutorial. We treat in the next four sections four different optimal stopping problems with our method. The first two are from sequential statistics, the other two from option pricing. Each section is self-contained and can be read separately. In all examples beside the first one we point out the use of Itô’s calculus. It is used to handle the drift terms of the processes under consideration. Either one tries to derive the function \( g \) as in the disruption problem (Section 3) or one wants to find the appropriate (local) martingale \( M \) (Sections 4 and 5).

Finally, in Section 6, we relate the B-L approach to Jamshidian’s recent results on a certain multiplicative minimax duality in optimal stopping and state some new results.

2 The repeated significance test

As a first example we consider a sequential testing problem that admits an explicit solution. Historically it was the first example for the B-L approach. For details, the reader is referred to [14, Ch. II, Sec. 5] or [15]. The exposition in this section is close to that given there, but here we treat also the case \( r \mu^2 > x^* \).

Suppose that we observe sequentially a continuous process \( W = (W_t)_{t \in [0, \infty)} \), and consider the statistical model \( (P_\theta; \theta \in \mathbb{R}) \) such that under \( P_\theta \), the process \( W \) is a Brownian motion with drift \( \theta t \). Consider the sequential testing problem \( H_0: \theta < 0 \) versus \( H_1: \theta > 0 \).

Let us put a normal prior \( G \) on \( \Theta = \mathbb{R} \setminus \{0\} \) with mean \( \mu \) and variance \( r^{-1} \), where \( r > 0 \). As sampling costs we choose \( c \theta^2, c > 0 \), which means that the sampling costs depend on the underlying parameter. Denote by \( (\mathcal{F}_t) \) the completed natural filtration of \( W \), and for any stopping time \( T \), define a final decision rule as an \( \mathcal{F}_T \)-measurable random variable \( \delta_T \) taking values 0 and 1, where \( \delta_T = 0 \) means that \( \delta_T \) accepts \( H_0 \) and \( \delta_T = 1 \) means that \( \delta_T \) accepts \( H_1 \). For a stopping time \( T \) and a final decision rule \( \delta_T \), the risk is then given
by

\[ R(T, \delta_T) = \int_{-\infty}^{0} (P_{\theta}\{\delta_T \text{ rejects } H_0\} + c\theta^2E_0T) \ G(d\theta) \\
+ \int_{0}^{\infty} (P_{\theta}\{\delta_T \text{ rejects } H_1\} + c\theta^2E_0T) \ G(d\theta). \quad (2.1) \]

The task is to find a pair \((T^*, \delta^*_T)\) which minimizes the risk:

\[ R(T^*, \delta^*_T) = \min_{(T, \delta)} R(T, \delta). \]

Let \(\Phi\) denote the distribution function of the standard normal distribution and \(\varphi\) the corresponding density. A straightforward calculation implies that the posterior on \(\Theta\) given \(\mathcal{F}_t\) is normal with mean \(W_t + r\mu\) and variance \(\frac{1}{T+r}\). Let us use the notation \(G_{W_t,t} = N\left(W_t + r\mu, \frac{1}{T+r}\right)\) and \(Q = \int_{\Theta} P_{\theta}G(d\theta)\). Clearly, if \(Q(T = \infty) > 0\), then for any final decision rule \(\delta_T, R(T, \delta_T) = \infty\). Therefore, we consider in the sequel only stopping times \(T\) such that \(Q(T < \infty) = 1\).

Let us calculate the part of the risk corresponding to the error probabilities:

\[
\begin{align*}
&\int_{-\infty}^{0} P_{\theta}\{\delta_T \text{ rejects } H_0\} \ G(d\theta) + \int_{0}^{\infty} P_{\theta}\{\delta_T \text{ rejects } H_1\} \ G(d\theta) \\
= &\int G_{W_T,T}(-\infty, 0)1_{\{\delta_T=1\}} \ dQ + \int G_{W_T,T}(0, \infty)1_{\{\delta_T=0\}} \ dQ \\
\geq &\int \min \left(G_{W_T,T}(-\infty, 0), G_{W_T,T}(0, \infty)\right) \ dQ \\
= &\int \Phi \left(-\frac{|W_T + r\mu|}{\sqrt{T+r}}\right) \ dQ.
\end{align*}
\]

It follows that for any stopping time \(T\), the final decision rule \(\delta^*_T = 1_{\{W_T+r\mu>0\}}\) is optimal because the inequality above turns into equality.

Now let us calculate the part of the risk corresponding to the sampling costs:

\[
\begin{align*}
&\int \theta^2E_0T \ G(d\theta) = \int T \left(\int \theta^2G_{W_T,T}(d\theta)\right) \ dQ \\
= &\int T \left(\frac{(W_T + r\mu)^2}{(T+r)^2} + \frac{1}{T+r}\right) \ dQ \\
= &\int (T+r) \left(\frac{(W_T + r\mu)^2}{(T+r)^2} + \frac{1}{T+r}\right) \ dQ - (r\mu^2 + 1) \\
= &\int \frac{(W_T + r\mu)^2}{T+r} \ dQ - r\mu^2.
\end{align*}
\]
Thus we obtain for the risk (2.1) with the final decision rule $\delta^*_T$,

$$R(T, \delta^*_T) = \int g \left( \frac{(W_T + r\mu)^2}{T + r} \right) dQ,$$

where $g(x) = \Phi(-\sqrt{x}) + cx - cr\mu^2$. It is easy to check that the function $g(x)$, $x \geq 0$, is convex and has a unique minimum at a point $x^*$ that is the unique solution of the equation $\varphi(\sqrt{x})/\sqrt{x} = 2c$.

**Theorem 2.1 (Lerche)** Let $T^* = \inf \{ t \geq 0 \mid \frac{(W_t + r\mu)^2}{t + r} \geq x^* \}$ (as usual $\inf \emptyset = \infty$). Then $(T^*, \delta^*_T)$ minimizes the risk (2.1).

**Proof:** Let $r\mu^2 \leq x^*$. Then by (2.2),

$$R(T, \delta^*_T) = \int g \left( \frac{(W_T + r\mu)^2}{T + r} \right) dQ \geq g(x^*) = R(T^*, \delta^*_T),$$

since $Q(T^* < \infty) = 1$. This proves the first part.

Let $r\mu^2 > x^*$. Since in this case $T^* \equiv 0$, we need only to prove that for any stopping time $T$ such that $Q(T < \infty) = 1$, we have $R(T, \delta^*_T) \geq g(r\mu^2)$.

To see this, we consider the $Q$-martingale

$$N_t = \frac{dP_0}{dQ} \bigg|_{\mathcal{F}_t} = \sqrt{\frac{t + r}{r}} \exp \left\{ -\frac{(W_t + r\mu)^2}{2(t + r)} + \frac{r\mu^2}{2} \right\},$$

where $P_0$ denotes the measure of Brownian motion without drift. Since $h(x) = 2\log \frac{1}{x}$ is convex, $h(N_t) = \frac{(W_t + r\mu)^2}{t + r} - r\mu^2 - \log \frac{t + r}{r}$ is a $Q$-submartingale. Hence

$$Z_t = \frac{(W_t + r\mu)^2}{t + r}$$

is a $Q$-submartingale. Since $g$ is convex on $[0, \infty)$ and increasing on $[x^*, \infty)$, we have for any bounded stopping time $T$,

$$R(T, \delta^*_T) = E_Q g(Z_T) \geq g(E_Q Z_T) \geq g(Z_0) = g(r\mu^2). \quad (2.3)$$

Now let $T$ be an arbitrary stopping time such that $Q(T < \infty) = 1$. We need only to prove that $R(T \wedge n, \delta^*_T \wedge n) \to R(T, \delta_T)$ as $n \to \infty$ since (2.3) holds for the stopping times $T \wedge n$. And this follows from the representation

$$R(T, \delta_T) = E_Q \Phi \left( -\frac{|W_T + r\mu|}{\sqrt{T + r}} \right) + c \int \theta^2 E_\theta T G(d\theta). \quad (2.4)$$
Indeed, the second term of the right-hand side of (2.4) for $T \wedge n$ converges to that for $T$ by the monotone convergence theorem; and the first term of the right-hand side of (2.4) for $T \wedge n$ converges to that for $T$ by the dominated convergence theorem.

\[ \square \]

3 The disruption problem

This problem was first treated in the dissertation of Shiryaev [20, 21] via the free boundary approach (for more details, see also [22]–[24]). It is the first optimality result for a sequential change point problem. Here we treat the disruption problem with the B-L approach. The derivation given here is a part of Beibel’s dissertation written under the supervision of the first author (see [1]). It turns out that many ideas necessary to solve this problem via the B-L approach are already present in the works of Shiryaev mentioned above. In the corresponding places, we shall refer to [23] and [24].

Suppose that we observe sequentially a process $W = (W_t; t \geq 0)$ which is given by the formula $W_t = B_t + \theta(t - \tau)^+ + \xi$, where $B = (B_t; t \geq 0)$ is a standard Brownian motion, $\theta$ is a positive constant, and $\tau$ is a random time with distribution $p_0 \delta_0 + (1 - p_0)F$. Here $0 \leq p_0 < 1$ and $F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$, where $\lambda > 0$. It is assumed that $\theta$, $p_0$, and $\lambda$ are known; $B$ and $\tau$ are unobservable and independent of each other. Denote by $(F_t)$ the completed natural filtration of $W$ and by $P$ the underlying measure. For a stopping time $T$, define the risk by the formula

$$R(T) = P(T < \tau) + cE(T - \tau)^+.$$  

The task is to find a stopping time $T^*$ for which $R(T^*) = \min_T R(T)$ holds.

**Theorem 3.1 (Shiryaev)** Consider the process $\pi_t = P(\tau \leq t \mid F_t)$, $t \geq 0$. Let $p^*$ denote the unique solution of $G'(p) = 1$, where $G$ is a solution of the equation

$$\frac{\theta^2}{2} x^2 (1-x)^2 G''(x) + \lambda (1-x) G'(x) = cx, \quad 0 < x < 1$$

such that $G'(0+) = 0$ (any two such solutions $G$ differ by a constant, so $p^*$ is defined correctly). Then the optimal stopping time $T^*$ is given by the formula

$$T^* = \inf \{ t \geq 0 \mid \pi_t \geq p^* \}.$$  

(3.1)
where as usual \( \inf \mathcal{O} = \infty \).

Before proving this theorem with our method let us recall some basic facts about the process \( \pi = (\pi_t; t \geq 0) \) (see [23] or [24]):

- With \( L_t = \exp\{\theta W_t - \theta^2/2\} \), we have
  \[
  \pi_t = \frac{\varphi_t}{e^{-\lambda t} + \varphi_t}, \text{ where } \varphi_t = \frac{p}{1-p} L_t + \int_0^t \frac{L_t \lambda e^{-\lambda s}}{L_s} ds.
  \]

- \( \pi \) is a diffusion process with
  \[
  d\pi_t = \lambda(1 - \pi_t)dt + \theta \pi_t (1 - \pi_t) d\overline{W}_t, \quad \pi_0 = p_0,
  \]
  where \( \overline{W} \) is an \( (\mathcal{F}_t) \)-Brownian motion given by \( \overline{W}_t = W_t - \theta \int_0^t \pi_s ds \) (in particular, \( \overline{W} \) is an observable process).

**Proof:** It is enough to consider stopping times \( T \) with \( ET < \infty \). Below we represent the risk as

\[
R(T) = E g^*(\pi_T) \tag{3.2}
\]

for stopping times \( T \) with \( ET < \infty \), where the function \( g^*(x), x \in [0,1] \), has a unique minimum attained at \( p^* \). Since \( \pi_0 = p_0 \) and \( \pi_t \to 1 \) a.s. as \( t \to \infty \), the statement of the theorem follows in the case \( p_0 = p^* \) provided \( ET^* < \infty \). And \( ET^* < \infty \) is proved below. Finally, we shall separately consider the case \( p_0 > p^* \).

To see that (3.2) holds, we note that

\[
R(T) = P(T < \tau) + c E(T - \tau)^+ = E \left[ (1 - \pi_T) + c \int_0^T \pi_s ds \right].
\]

We write \( g(x) = (1 - x) + G(x) - G(p_0) \) and try to find an appropriate \( G \) by applying Itô’s formula to \( G(\pi_t) \). Then

\[
dG(\pi_t) = G'(\pi_t)d\pi_t + \frac{1}{2} G''(\pi_t)d(\pi_t)^2
\]

\[
= G'(\pi_t)[\lambda(1 - \pi_t)dt + \theta \pi_t (1 - \pi_t) d\overline{W}_t]
+ \frac{1}{2} G''(\pi_t) \theta^2 \pi_t^2(1 - \pi_t)^2 dt
\]

\[
= G'(\pi_t) \theta \pi_t (1 - \pi_t) d\overline{W}_t
+ \left[ \frac{1}{2} G''(\pi_t) \theta^2 \pi_t^2(1 - \pi_t)^2 + G'(\pi_t) \lambda(1 - \pi_t) \right] dt.
\]
Therefore, if $G$ satisfies the equation
\[ \frac{\theta^2}{2} x^2 (1 - x)^2 G''(x) + \lambda (1 - x) G'(x) = c x, \quad 0 < x < 1, \] (3.3)
then we hope to have the representation
\[ G(p_1) - G(p_0) = c \int_0^t \pi_s ds + M_t, \] (3.4)
where
\[ M_t = \theta \int_0^t \pi_s (1 - \pi_s) G''(\pi_s) d\bar{W}_s \] (3.5)
is a local martingale starting from 0. (Singularity of the coefficients of (3.3) at $x = 0$ and $x = 1$ makes us consider (3.3) only on the open interval $(0, 1)$, and this results in the fact that we should check (3.4) and (3.5) after we take an appropriate solution of (3.3).) Putting $u(x) = G'(x)$, rewriting (3.3) in the form
\[ u'(x) = \frac{c x - \lambda (1 - x) u(x)}{\frac{\theta^2}{2} x^2 (1 - x)^2}, \quad 0 < x < 1, \] (3.6)
and analysing the integral curves of (3.6), one can verify (see [23] or [24]) that all the integral curves of (3.6) are divided into three disjoint classes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ with the following properties:

- for any $u \in \mathcal{A}$, $u(0+) = -\infty$, $u(1-) = \infty$;
- for any $u \in \mathcal{B}$, $u(0+) = \infty$, $u(1-) = \infty$;
- for any $u \in \mathcal{A}$ and $v \in \mathcal{B}$, $u$ lies below $v$;
- the class $\mathcal{C}$ defined as the rest of the integral curves of (3.6) consists of exactly one curve $u^*$ that is the separatrix between $\mathcal{A}$ and $\mathcal{B}$;
- $u^*(0+) = 0$ (and by definition we put $u^*(0) = 0$, $u^*$ is strictly increasing, $u^*(1-) = \infty$, the function $(1 - x)u^*(x)$ is bounded on $[0, 1)$, and $\int_0^1 u^*(x) dx = \infty$.

Now consider any function $G^*$ such that
\[ G^{**}(x) = u^*(x), \quad 0 \leq x < 1. \]
It follows from the properties of the function \( g^* \) that it is strictly increasing and strictly convex on \([0, 1)\), \( G''(0) = G''(0+) = 0 \), and \( G'(1-) = \infty \). Since the process \( \pi \) never hits 1 (see the formula for \( \pi \) before the proof), we can apply the Itô-Tanaka formula (see [16, Ch. VI, Sec. 1]) to \( G^*(\pi_t) \). By existence of \( G'' \) on \((0, 1)\) and continuity of \( G'' \) at 0, this formula reduces here to Itô’s formula. Hence (3.4) and (3.5) for \( G^* \) are proved.

Now put \( g^*(x) = (1 - x) + G^*(x) - G^*(p_0) \) and consider any stopping time \( T \) such that \( ET < \infty \). By (3.5) and boundedness of \( x(1 - x)G''(x) \), \( 0 \leq x < 1 \), we have \( E(M_T) < \infty \), hence \( EM_T = 0 \). By (3.4), we have \( E(G^*(\pi_T) - G^*(p_0)) = cE \int_0^T \pi_s \, ds \) and obtain (3.2).

It follows from the properties of the function \( G^* \) stated above that \( g^* \) is strictly convex on \([0, 1)\), strictly decreasing on \([0, p^*] \), strictly increasing to \( \infty \) on \([p^*, 1) \), where \( p^* \) is the unique solution of the equation \( G''(p) = 1 \). Now let \( p_0 \leq p^* \). As it was noted at the beginning of the proof, we need only to prove that \( ET^* < \infty \) (the stopping time \( T^* \) is defined in (3.1)), and this is a consequence of the following inequalities:

\[
P(T^* > t) \leq P(1 - \pi_t > 1 - p^*) \leq \frac{E(1 - \pi_t)}{1 - p^*} = \frac{(1 - p_0)e^{-\lambda t}}{1 - p^*}.
\]

Finally, let \( p_0 > p^* \). Then \( T^* \equiv 0 \). Since \( \pi \) is a bounded submartingale, \( g \) is convex, and \( g \) is increasing on \([p^*, 1) \), we have for any stopping time \( T \) such that \( ET < \infty \),

\[
R(T) = Eg(\pi_T) \geq g(E\pi_T) \geq g(p_0) = R(T^*).
\]

\[\square\]

**Remark 3.2** It is interesting to note that one needs to solve an equation similar to (3.3) for the value function when one solves the disruption problem via the free boundary approach (the right-hand side is \(-cx\) instead of \(cx\); see [23] or [24]). But unlike the situation here, the value function satisfies this equation only in the continuation region that is determined with the help of the continuous and smooth fit conditions. And here the function \( G^* \) satisfies (3.3) on the whole interval \((0, 1)\), while the stopping boundary is determined through minimization of the function \( g^* \). The smooth fit condition \( G'(p^*) = 1 \) is here a direct consequence of the minimum property of \( p^* \).
4 Perpetual Russian option

The perpetual Russian option was introduced in the papers [18] and [19] of Shepp and Shiryaev. See also the monograph [25, Ch. VIII, Sec. 2d] for a more detailed discussion closely related to the presentation given here. Below we treat this problem with the B-L approach.

Let

\[ X_t = \exp\{\sigma B_t + (\mu - \sigma^2/2)t\} \]

for \( t \geq 0 \), where \( B \) is a standard Brownian motion with \( B_0 = 0 \), \( \sigma > 0 \), and \( \mu \in \mathbb{R} \). Consider the completed filtration \((\mathcal{F}_t)\) generated by \( B \). Let \( S_t = \max_{0 \leq s \leq t} X_s \) denote the running maximum of \( X \). We shall consider the following problem: Find a stopping time \( T \) that maximizes

\[ E(e^{rT}S_T) \]

over all stopping times \( T \), where \( r > \max(\mu, 0) \) (on the set \( \{T = \infty\} \) we define \( e^{rT}S_T = \lim_{t \to \infty} e^{rt}S_t = 0 \)). Note that the other natural case \( r > 0 \) is less interesting here because in this case, \( \sup_T E(e^{rT}S_T) = \infty \).

Theorem 4.1 (Shepp, Shiryaev)

Let \( \gamma_{1,2} = -\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\frac{2\mu}{\sigma^2} + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2} \) and \( \alpha = (1 - \frac{1}{1/\gamma_2 - 1/\gamma_1})^{1/(\gamma_2 - \gamma_1)} \). Then

\[ \max_T E(e^{-rT}S_T) = E(e^{-rT^*}S_{T^*}), \]

where \( T^* = \inf\{t \geq 0 \mid S_t \geq \alpha \} \) and as usual \( \inf \emptyset = \infty \). (Note that \( \gamma_1 < 0 \) and \( \gamma_2 > 1 \), hence \( \alpha > 1 \).)

Proof: The continuous semimartingale \( X_t/S_t \) has the stochastic differential

\[ d(X_t/S_t) = \frac{1}{S_t} dX_t - \frac{X_t}{S_t^2} dS_t \]  

(4.1)

since \( S \) has increasing paths. For sufficiently smooth functions \( h \), we therefore obtain

\[ e^{-rT}S_T h\left(\frac{X_t}{S_t}\right) = h(1) + \int_0^t e^{-ru}S_u \left\{ -rh\left(\frac{X_u}{S_u}\right) + \mu\left(\frac{X_u}{S_u}\right)h'\left(\frac{X_u}{S_u}\right) + \frac{\sigma^2}{2}\left(\frac{X_u}{S_u}\right)^2 h''\left(\frac{X_u}{S_u}\right) \right\} du \]

\[ + \int_0^t e^{-ru} \left\{ -\left(\frac{X_u}{S_u}\right)h\left(\frac{X_u}{S_u}\right) + h\left(\frac{X_u}{S_u}\right) \right\} dS_u \]

\[ + \sigma \int_0^t e^{-ru}X_u h'\left(\frac{X_u}{S_u}\right) dB_u. \]
The process $S$ is flat off the set $\{ t \mid S_t = X_t \}$. Therefore

$$
\int_0^t e^{-ru} \left\{ - \left( \frac{X_u}{S_u} \right) h' \left( \frac{X_u}{S_u} \right) + h \left( \frac{X_u}{S_u} \right) \right\} dS_u
= \int_0^t e^{-ru} \{ -h'(1) + h(1) \} dS_u.
$$

Hence for sufficiently smooth functions $h$ the process $e^{-rt}S_t h \left( \frac{X_t}{S_t} \right)$ is a local martingale if $h$ satisfies

$$
0 = -rh(x) + \mu x h'(x) + \frac{\sigma^2}{2} x^2 h''(x) \quad \text{for all } x \in (0, 1),
0 = h'(1) - h(1).
$$

Solutions of these equations are multiples of the function $\gamma_2(\alpha x)^{\gamma_1} - \gamma_1(\alpha x)^{\gamma_2}$ (see [18]), where $\gamma_1$, $\gamma_2$, and $\alpha$ are as above. Let us take the solution $h(x)$ such that $h(1) = 1$, i.e.,

$$
h(x) = \frac{1}{\gamma_2 \alpha^{\gamma_1} - \gamma_1 \alpha^{\gamma_2}} \left( \gamma_2(\alpha x)^{\gamma_1} - \gamma_1(\alpha x)^{\gamma_2} \right).
$$

It is easy to see that

$$
\min_{0 < x \leq 1} h(x) = h \left( \alpha^{-1} \right) > 0
$$

holds. Since $M_t = e^{-rt}S_t h \left( \frac{X_t}{S_t} \right)$ is a positive local martingale (hence supermartingale) and $M_0 = 1$, we obtain

$$
E(e^{-rT}S_T) = E[e^{-rT}S_T 1(T < \infty)] = E \left[ \frac{M_T 1(T < \infty)}{h \left( \frac{X_T}{S_T} \right)} \right]
\leq \frac{1}{h(\alpha^{-1})} E[M_T 1(T < \infty)] \leq \frac{1}{h(\alpha^{-1})} EM_T \leq \frac{1}{h(\alpha^{-1})}.
$$

Now let us recall that $P(T^* < \infty) = 1$ by just repeating the beautiful
argument in [18]. Indeed, for any $n \in \mathbb{N}$, we have

$$P(T^* > n) = P \left( \forall 0 \leq u \leq t \leq n, \sigma(B_t - B_u) + (\mu - \sigma^2/2)(t - u) > \log \alpha^{-1} \right)$$

$$\leq P \left( \forall 0 \leq u \leq t \leq 1, \sigma(B_t - B_u) > \log \alpha^{-1} - \left| \mu - \frac{\sigma^2}{2} \right| \right)^n$$

which tends to zero as $n \to \infty$ no matter what the sign of $\mu - \sigma^2/2$ is. Thus, $P(T^* < \infty) = 1$ and so

$$h \left( \frac{X_{T^*}}{S_{T^*}} \right) = h(\alpha^{-1}).$$

Therefore it is only left to show that $\overline{EM}_{T^*} = 1$. A sufficient condition for this equality is

$$E \sup_{0 \leq t \leq T^*} e^{-r t} S_t h \left( \frac{X_t}{S_t} \right) < \infty.$$

For $0 \leq t \leq T^*$ we have

$$0 < \frac{1}{\alpha} \leq \frac{X_t}{S_t} \leq 1.$$

The continuous function $h$ is bounded on the compact interval $[\frac{1}{\alpha}, 1]$. Hence it is sufficient to show that

$$E \sup_{0 \leq t \leq T^*} e^{-r t} S_t < \infty.$$

For all $t \geq 0$, it holds

$$e^{-r t} \sup_{0 \leq u \leq t} X_u \leq \sup_{0 \leq u \leq t} e^{-r u} X_u$$

and hence

$$\sup_{0 \leq t < \infty} e^{-r t} S_t \leq \sup_{0 \leq t < \infty} \sup_{0 \leq u \leq t} e^{-r u} X_u = \sup_{0 \leq u < \infty} e^{-r u} X_u.$$

This yields

$$E \sup_{0 \leq t \leq T^*} e^{-r t} S_t \leq E \sup_{0 \leq u < \infty} e^{-r u} X_u < \infty.$$
where the finiteness of the last expectation can be easily verified.

The argument in the proof of Theorem 4.1 can also be used to discuss lookback options. This means that we consider the following problem: Find a stopping time $T^*$ that maximizes $E[e^{-rT}(S_T - X_T)]$, where $r > \mu$ (on the set $\{T = \infty\}$ we define $e^{-rT}(S_T - X_T) = \lim_{t \to \infty} e^{-rt}(S_t - X_t) = 0$). Using the preceding calculations, we have the representation and the estimate

$$E[e^{-rT}(S_T - X_T)] = E\left[\frac{1 - \frac{X_t}{S_t}}{h}\left(\frac{X_T}{S_T}\right)M_T I(T < \infty)\right] \leq \frac{1 - x^*}{h(x^*)},$$

where $x^* = \arg\max\{(1 - x)/h(x) \mid x \in (0, 1]\}$. Then $T^* = \inf\{t \geq 0 \mid X_t/S_t \leq x^*\}$ is optimal.

**Remark 4.2 (Structure of $M$)** The local martingale $M$ appearing in the proof above satisfies

$$M_t = 1 + \sigma \int_0^t e^{-ru} X_u h'(\frac{X_u}{S_u}) dB_u = 1 + \sigma \int_0^t M_u \frac{X_u}{S_u} h'(\frac{X_u}{S_u}) dB_u.$$

The function $\Lambda$ with

$$\Lambda(x) = x h'(x) \frac{h(x)}{\gamma_1 \gamma_2 \gamma_1 \gamma_2 (ax)^{\gamma_1} - (ax)^{\gamma_2}} \gamma_2 (ax)^{\gamma_1} - \gamma_1 (ax)^{\gamma_2}$$

is bounded on $(0, 1]$. Therefore the stochastic integral $Y$ with

$$Y_t = \sigma \int_0^t \Lambda \left(\frac{X_u}{S_u}\right) dB_u$$

is well-defined. The process $M$ satisfies the Doléans equation

$$dM_t = M_t dY_t$$

with $M_0 = 1$. Hence we have

$$M_t = \exp\left\{Y_t - \frac{1}{2}(Y)_t\right\}$$

$$= \exp\left\{\sigma \int_0^t \Lambda \left(\frac{X_u}{S_u}\right) dB_u - \frac{\sigma^2}{2} \int_0^t \Lambda^2 \left(\frac{X_u}{S_u}\right) du\right\}. \quad (4.2)$$
Since \( \Lambda(\frac{X_u}{S_u}) \) is a bounded process, the Novikov condition implies that the local martingale \( M \) is moreover a martingale.

Further,
\[
\langle Y \rangle_\infty = \sigma^2 \int_0^\infty \Lambda^2 \left( \frac{X_u}{S_u} \right) \, du = \infty.
\]
By the Dambis-Dubins-Schwarz theorem, \( \lim \inf_{t \to \infty} Y_t = -\infty \). Hence it follows from (4.2) that
\[
\lim_{t \to \infty} M_t = \lim \inf_{t \to \infty} M_t = 0
\]
(\( M \) has a limit as \( t \to \infty \) because it is a positive martingale). Therefore, \( M \) is not a uniformly integrable martingale. This will be used in Section 6.

5 Perpetual integral option

This example is related to the preceding one. The topic was first treated by Kramkov and Mordecki [13]. The exposition below is close to that in [13], but we apply the B-L approach instead of writing down the verification theorem. Some similar arguments are used in [4] for another problem.

Let \( X_t = \exp\{\sigma B_t + (\mu - \sigma^2/2)t\} \) for \( t \geq 0 \), where \( B \) is a standard Brownian motion with \( B_0 = 0 \), \( \sigma > 0 \), and \( \mu \in \mathbb{R} \). Consider the completed filtration \( (\mathcal{F}_t) \) generated by \( B \). Let \( A_t = \int_0^t X_u \, du \). We shall consider the following problem: Find a stopping time \( T^* \) that maximizing \( E(e^{-rt}A_T) \) over all stopping times \( T \), where \( r > \max(\mu, 0) \) (on the set \( \{T = \infty\} \) we define \( e^{-rt}A_T = \lim_{t \to \infty} e^{-rt}A_t = 0 \)). Note that the other natural case \( \mu \geq r > 0 \) is less interesting here because in this case, \( \sup_T E(e^{-rt}A_T) = \infty \).

Theorem 5.1 (Kramkov, Mordecki)

Let \( \gamma_{1,2} = \left( \frac{\mu}{\sigma^2} + \frac{1}{2} \right) \mp \sqrt{\frac{2(\tau - \mu)}{\sigma^2} + \left( \frac{\mu}{\sigma^2} + \frac{1}{2} \right)^2} \) and note that \( \gamma_1 < 0 \) and \( \gamma_2 > 1 \). Define the function
\[
g(x) = \frac{(\sigma^2/2)^{\gamma_1}}{\Gamma(-\gamma_1)} \int_0^\infty \exp \left\{ -\frac{2y}{\sigma^2} \right\} y^{-(\gamma_1+1)}(1 + xy)^{\gamma_2} \, dy, \quad x \geq 0. \tag{5.1}
\]
and let \( x^* \) be the unique root of the equation \( xg'(x) = g(x) \). Then the optimal stopping time is given by
\[
T^* = \inf \left\{ t \geq 0 \mid \frac{A_t}{X_t} \geq x^* \right\}.
\]
where as usual \( \inf \Omega = \infty \).

**Remark 5.2** The functions \( g(x) \) here and \( u(x) \) in Kramkov and Mordecki [13] are multiples of each other: \( g(x) = \frac{(\sigma^2/2)^{x}}{x((x-\gamma)^{1}}u(x) \), \( g \) being chosen such that \( g(0) = 1 \).

Let us now prove the result of Kramkov and Mordecki with our technique.

**Proof of Theorem 5.1:** At first let us rewrite the expected reward:

\[
E(e^{-rT} A_T) = E[e^{-rT} 1(T < \infty)]
\]

\[
= E \left[ e^{-rT} A_T X_T \exp \left\{ \sigma B_T - \frac{\sigma^2}{2} T \right\} e^{\mu T} 1(T < \infty) \right]
\]

\[
= E \left[ e^{-(r-\mu)T} A_T X_T \exp \left\{ \sigma B_T - \frac{\sigma^2}{2} T \right\} 1(T < \infty) \right]
\]

\[
= \tilde{E} \left[ e^{-\lambda T} A_T X_T 1(T < \infty) \right],
\]

where we set \( \lambda = r - \mu \) and introduce the measure \( \tilde{P} \) as in [13] by the formula

\[
\frac{d\tilde{P}}{dP} |_{\mathcal{F}_t} = \exp \{ \sigma B_t - (\sigma^2/2)t \}, \quad t \geq 0.
\]

Below in the proof, we shall work only under the measure \( \tilde{P} \). Note that \( \tilde{B}_t = B_t - \sigma t \) is a \( \tilde{P} \)-Brownian motion.

Now we denote \( Y_t = \frac{A_t}{X_t} \). So, our task is to maximize \( \tilde{E}[e^{-\lambda T} 1(T < \infty)] \) over all stopping times \( T \). Let us try to find a martingale to get rid of the term \( e^{-\lambda T} \).

Under the measure \( \tilde{P} \), we have

\[
dX_t = X_t((\mu + \sigma^2) dt + \sigma dB_t).
\]

By Itô’s formula,

\[
d\frac{1}{X_t} = \frac{1}{X_t^2} dX_t + \frac{1}{X_t^3} d\langle X \rangle_t = \frac{\mu}{X_t} dt - \frac{\sigma}{X_t} dB_t.
\]

Hence

\[
dY_t = \frac{1}{X_t} dA_t + A_t \frac{1}{X_t} = (1 - \mu Y_t) dt - \sigma Y_t dB_t.
\]

Then for a smooth function \( h \), it holds

\[
d[h(Y_t)] = \left[ (1 - \mu Y_t) h'(Y_t) + \frac{\sigma^2}{2} Y_t^2 h''(Y_t) \right] dt - \sigma Y_t h'(Y_t) dB_t.
\]
Therefore,
\[
    d[e^{-\lambda t}h(Y_t)] = -\lambda e^{-\lambda t}h(Y_t) dt + e^{-\lambda t} d[h(Y_t)] \\
    = e^{-\lambda t} \left[ -\lambda h(Y_t) + (1 - \mu Y_t)h'(Y_t) + \frac{\sigma^2}{2} Y_t^2 h''(Y_t) \right] dt \\
    + e^{-\lambda t} [-\sigma Y_t h'(Y_t)] d\tilde{B}_t.
\] (5.2)

The process \(e^{-\lambda t}h(Y_t)\) is a \(\tilde{P}\)-local martingale if and only if the drift term in (5.2) vanishes. Thus we obtain the following differential equation for \(h\):
\[
    \frac{\sigma^2}{2} x^2 h''(x) + (1 - \mu x) h'(x) - \lambda h(x) = 0, \quad x \geq 0.
\] (5.3)

The function \(g(x)\) introduced in (5.1) is a solution of this equation satisfying the following conditions: \(g(0) = 1\), \(g\) is strictly increasing and strictly convex, \(g(x)/x \to \infty\) as \(x \to \infty\) (see [13]). Hence the function \(x/g(x), x \geq 0\), has a unique maximum at a point \(x^*\) that is the unique root of the equation \(xg'(x) = g(x)\).

Now consider the positive \(\tilde{P}\)-local martingale (hence \(\tilde{P}\)-supermartingale) \(M_t = e^{-\lambda t}g(Y_t)\). We get
\[
    \tilde{E}[e^{-\lambda T} Y_T 1(T < \infty)] = \tilde{E} \left[ \frac{Y_T}{g(Y_T)} M_T 1(T < \infty) \right] \leq \frac{x^*}{g(x^*)}.
\] (5.4)

Now set \(T^* = \inf\{t \geq 0 \mid Y_t \geq x^*\}\). Then according to [13]
\[
    \tilde{P}(T^* < \infty) = 1.
\]

We get equality in (5.4) for \(T^*\) because \(\tilde{E} M_{T^*} = 1\), which follows from the uniform boundedness of the process \((M_{T^*}: T^* \leq x^*)\).

**Remark 5.3** Equation (5.3) is just the same as the equation for the value function arising in the free boundary approach (see [13]). But unlike the situation here, the value function satisfies it only in the continuation region that is determined with the help of the continuous and smooth fit conditions. And here the function \(g\) satisfies (5.3) on the whole interval \([0, \infty)\), while the stopping boundary is determined through maximization of the function \(x/g(x)\) (cf. with Remark 3.2).

**Remark 5.4** It can be shown like it was done in Remark 4.2 that the local martingale \(M\) is moreover a martingale but it is not uniformly integrable. This will be used in Section 6.
6 Some remarks on the relation to Jamshidian’s multiplicative minimax duality

In this section, we discuss the relation of the B-L approach to some recent developments on minimax duality in optimal stopping. Davis and Karatzas [8], Rogers [17], and Haugh and Kogan [10] introduced a certain additive minimax duality in optimal stopping and used it for Monte Carlo pricing of American and Bermudan options. Recently Jamshidian [11, 12] proposed a multiplicative minimax duality, which could also be used for similar purposes. Here we discuss some aspects of this duality that are related to the B-L approach.

In Subsection 1, we recall some basic facts about Jamshidian’s duality. In Subsection 2, we formulate our setting and pose some questions on the matter. In Subsection 3, we state the answers in the framework of a slightly modified multiplicative duality. In Subsection 4, we state the answers in the framework of Jamshidian’s duality. In this section, we describe some basic facts. The results are stated without proofs. In more detail (and with proofs) these and related questions will be discussed in a forthcoming paper of the authors.

Below we consider a stochastic basis \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, P) \) satisfying the usual conditions. For simplicity we always assume that the \( \sigma \)-field \( \mathcal{F}_0 \) is \( P \)-trivial. Let \( \mathcal{M} \) (resp., \( \mathcal{M}_f \)) denote the class of all (resp., \( P \)-finite) stopping times.

1. The multiplicative minimax duality mentioned above is the following result.

**Theorem 6.1 (Minimax duality; Jamshidian)** Let \( Z = (Z_t)_{t \in [0, \infty)} \) be a right-continuous adapted process of class \( D \) (i.e., the family of random variables \( Z_T \), where \( T \) are finite stopping times, is uniformly integrable). Set \( Z_\infty = \limsup_{t \to \infty} Z_t \) and suppose that \( Z_\infty \geq 0 \) a.s. Then

\[
\sup_{T \in \mathcal{M}} EZ_T = \sup_{T \in \mathcal{M}_f} EZ_T = \inf_{N \in \mathcal{C}^+} E \left( N_\infty \sup_{t \geq 0} \frac{Z_t}{N_t} \right), \tag{6.1}
\]

where \( \mathcal{C}^+ \) is the class of (right-continuous) uniformly integrable martingales \( N \) with strictly positive limit \( N_\infty \).

Note that if \( N \in \mathcal{C}^+ \), then \( N \) is strictly positive; therefore, the right-hand side of (6.1) is well defined. For the proof, see [12] (and note that the fact that
Let $Z$ be a right-continuous adapted process of class $(D)$. A **dominating martingale** for $Z$ is a uniformly integrable martingale $M$ such that sup$_{t \in [0, \infty)} (Z_t - M_t) = 0$ a.s. Jamshidian [12] proved that a uniformly integrable martingale $M$ dominates $Z$ if and only if $M \leq Z$ a.s. and $M_0 = \sup_{t \in [0, \infty)} EZ_T$.

Generally, there exist many dominating martingales for a given right-continuous adapted process $Z$ of class $(D)$ (see [12] for the construction of all of them). One of them can be constructed as follows. Let $V$ be the Snell envelope of $Z$. It is a right-continuous supermartingale of class $(D)$ (see [12] or also [9, App. 1]) and therefore has the Doob-Meyer decomposition $V = M - A$, $M_0 = V_0$, $A_0 = 0$, where $M$ is a uniformly integrable martingale. It is easy to verify that $M$ dominates $Z$.

The infimum at the right-hand side of (6.1) is attained at any uniformly integrable martingale $M$ that dominates $Z$ and at multiples of $M$ whenever $M_\infty > 0$ a.s. If $M$ is a dominating martingale for $Z$ but $P(M_\infty = 0) > 0$, then the sequence $N^{(n)} = M + 1/n \in C^+$ is a minimizing sequence of martingales for the right-hand side of (6.1).

2. The setting below is as follows. Let $Z = (Z_t)_{t \in [0, \infty)}$ be a continuous adapted process. Set $Z_\infty = \limsup_{t \to \infty} Z_t$. Assume that we are in the framework when the B-L approach works, i.e., the process $Z$ admits a decomposition

$$Z_t = g(X_t)M_t \quad a.s., \quad t \in [0, \infty), \quad (6.2)$$

where $X$ is a continuous adapted process, $M$ is a nonnegative local martingale starting from $M_0 = 1$ (hence it is a nonnegative supermartingale, hence it has a limit $M_\infty$), and max$_{x \in \mathbb{R}} g(x) = g(x^*) > 0$ for some $x^* \in \mathbb{R}$. Clearly, this implies that for any stopping time $T$, $EZ_T \leq g(x^*)$ (note that $Z_\infty \leq g(x^*)M_\infty$ a.s. because $Z_t \leq g(x^*)M_t$ a.s., $t \in [0, \infty)$). Now consider the stopping time

$$T^* = \inf \{ t \in [0, \infty) : X_t = x^* \}$$

(as usual $\inf \emptyset = \infty$) and suppose that the following conditions ($C_1$) and ($C_2$) are satisfied:

($C_1$) \quad $Z_\infty = g(x^*)M_\infty$ a.s. on the set $\{T^* = \infty\}$;
(C2) $EM_{T^*} = 1$.

Then $EZ_{T^*} = g(x^*)$ (moreover, it is easy to verify that (C1) and (C2) are satisfied if and only if $EZ_{T^*} = g(x^*)$). Therefore under conditions (C1) and (C2), we have

$$\sup_{T \in \mathcal{G}} EZ_T = g(x^*)$$

and $T^*$ is an optimal stopping time.

Here we described the B-L approach a bit more generally than in the introduction. In particular, we discuss what happens on the set $\{T^* = \infty\}$ (condition (C1)). This is reasonable because there exist natural optimal stopping problems, where $P(T^* = \infty) > 0$ for the optimal stopping time $T^*$ (see, for example, the problem of the perpetual American put option in [5]). Such problems will also be incorporated in the discussion below.

In the sequel, we are interested in the following two questions:

**Question 1.** How is the Snell envelope of $Z$ related to decomposition (6.2)?

**Question 2.** How is the (local) martingale $M$ related to a minimizing (uniformly integrable) martingale for the right-hand side of (6.1)?

3. It turns out that the answers to these questions become more transparent if we state the questions for the stopped process $Z_{T^*}$ rather than for $Z$. The answers are presented in the following slightly modified duality theorem.

**Theorem 6.2 (Minimax duality)** Suppose that we have decomposition (6.2), and conditions (C1) and (C2) hold. Assume additionally that the following condition is satisfied:

(C3) The process $(Z_{T^* \wedge t})_{t \in [0, \infty)}$ belongs to class (D).

Then

$$\sup_{T \in \mathcal{G}} EZ_T = \sup_{T \in \mathcal{G}} E[Z_T I(T < \infty)] = \sup_{T \in \mathcal{G}_f} EZ_T$$

$$= g(x^*) = \inf_{N \in \mathcal{C}^+} E \left( \sup_{t \geq 0} \frac{Z_{T^* \wedge t}}{N_{T^* \wedge t}} \right).$$

Further, the sequence $N^{(n)} = M^{T^*} + 1/n \in \mathcal{C}^+$ is a minimizing sequence for the right-hand side of (6.3). If $M^{T^*} > 0$ a.s., then $M^{T^*} \in \mathcal{C}^+$, and it is a minimizing martingale for the right-hand side of (6.3). Finally, $g(x^*)M^{T^*}$ is the Snell envelope of $Z^{T^*}$. 

19
We would like to note that Theorem 6.2 can be applied to many natural optimal stopping problems, where the B-L approach works. Indeed, if we are in this framework (i.e., we have decomposition (6.2), and conditions (C1) and (C2) are satisfied), then the additional assumption (C3) is not at all restrictive. If (C2) holds, (C3) follows from

\[(C4) \text{ The process } (g(X_{t \wedge T}))_{t \in [0, \infty)} \text{ is bounded from below.}\]

Condition (C4) turns out to be satisfied in many natural optimal stopping problems, where the B-L approach can be applied. For instance, it is easily seen that (C4) holds in all the examples considered in this paper (as far as the problems in Sections 2 and 3 are concerned, one should deal with the process \((-Z_t + \text{const})\) instead of \((Z_t)\) to turn from minimization to maximization and to ensure that \(g(x^*) > 0\).

Finally, let us point out that Theorem 6.1 cannot be applied to the problems of Sections 2 and 3 because \(Z\) does not belong to class \((D)\) there (and also \(Z_\infty = -\infty\)). This is the reason why we modified the duality.

4. Now we describe what kind of answers to the questions above can be given in the framework of duality (6.1). The answer to Question 1 is very simple:

**Proposition 6.3** The Snell envelope \(V_t = \text{ess sup}_{Z_t \geq t} E(Z_T | F_t), t \in [0, \infty),\) is well defined in the framework when the B-L approach works, and \(V_t \leq g(x^*)M_t\) a.s., \(t \in [0, \infty).\) One can take a version of \(V\) such that \(V = g(x^*)M\) on the stochastic interval \([0, T^*]\).

We note that it is not clear how to characterize the Snell envelope \(V\) after \(T^*\). Generally, in the setting of the B-L approach it is even not known whether \(V\) is a supermartingale ( \(g\) has not to be bounded from below). But in many examples this is the case.

The usual situation is that \(V < g(x^*)M\) on the stochastic interval \((T^*, \infty).\) The reason is that \(M\) is a martingale, \(V\) is a supermartingale, and usually \(V\) is a “strict supermartingale” after \(T^*\).

Let us turn to Question 2. It is easy to understand that if the conditions of Theorem 6.1 are satisfied (namely \(Z\) belongs to class \((D)\) and \(Z_\infty = \lim \sup_{t \to \infty} Z_t \geq 0\) a.s.) and \(M \in C^+\), then the infimum at the right-hand side of (6.1) is attained at \(M\). Moreover, if \(M\) is a uniformly integrable nonnegative martingale but not necessarily with strictly positive limit, then the sequence \(N^{(n)} = M + 1/n \in C^+\) is a minimizing sequence of martingales.
for the right-hand side of (6.1). (Generally, there exist other minimizing martingales as it was noted in Subsection 1. Clearly, their multiples are also minimizing. But here we are interested in $M$ as it was stated in Question 2.)

However, in many concrete examples, one of the following two variants holds:

**Variant A.** The conditions of Theorem 6.1 are not satisfied (and so we cannot guarantee (6.1) and speak about a minimizing martingale for its right-hand side);

**Variant B.** The conditions of Theorem 6.1 are satisfied, the nonnegative local martingale $M$ is a martingale, but it is not uniformly integrable.

Clearly, in the statistical examples considered above (Sections 2 and 3), Variant A holds. It can be proved (but requires some space) that in the examples arising in mathematical finance considered above (Sections 4 and 5), Variant B holds (a part of this statement is the content of Remarks 4.2 and 5.4).

Question 2 makes only sense in the situation of Variant B. In this situation, the first natural idea arising from the discussion above is that $M_{T^*}$ should be a minimizing martingale for the right-hand side of (6.1) provided $M_{T^*} > 0$ a.s. (and otherwise $N^{(n)} = M_{T^*} + 1/n \in C^+$ should be a minimizing sequence of martingales). But it can be verified that this is wrong for the problems of Sections 4 and 5. Such a situation seems to be quite usual.

Finally, let us state a positive result in this direction. It can be verified that the sequence $N^{(m)} = (M_{m,n}) \in C^+$ is a minimizing sequence of martingales for the right-hand side of (6.1) in the examples of Sections 4 and 5. And such a sequence appears to be minimizing in a number of other optimal stopping problems in the framework when the B-L approach works and Variant B holds.

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