ON A PROPERTY OF THE MOMENT AT WHICH
BROWNIAN MOTION ATTAINS ITS MAXIMUM
AND SOME OPTIMAL STOPPING PROBLEMS

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Let $B = (B_t)_{0 \leq t \leq 1}$ be a standard Brownian motion and $\theta$ be the moment at which $B$ attains its maximal value, i.e., $B_\theta = \max_{0 \leq t \leq 1} B_t$. Denote by $(\mathcal{F}^B_t)_{0 \leq t \leq 1}$ the filtration generated by $B$. We prove that for any $(\mathcal{F}^B_t)$-stopping time $\tau$ ($0 \leq \tau \leq 1$), the following equality holds:

$$E(B_\theta - B_\tau)^2 = E(\theta - \tau) + \frac{1}{\tau}.$$  

Together with the results of [1] this implies that the optimal stopping time $\tau^*$ in the problem

$$\inf_{\tau} E(\theta - \tau)$$

has the form

$$\tau^* = \inf \{0 \leq t \leq 1 : S_t - B_t \geq z^* \sqrt{1 - T} \},$$

where $S_t = \max_{0 \leq s \leq t} B_s$, $z^*$ is a unique positive root of the equation $4\Phi(z) - 2\varphi(z) - 3 = 0$, $\varphi(z)$ and $\Phi(z)$ are the density and the distribution function of a standard Gaussian random variable. Similarly, we solve the optimal stopping problems

$$\inf_{\tau \in \mathcal{G}_\alpha} E(\tau - \theta)^+$ \\ and \hspace{0.5cm} \inf_{\tau \in \mathcal{G}_\alpha} E(\tau - \theta)^-,$$

where $\mathcal{G}_\alpha = \{ \tau : E(\tau - \theta)^- \leq \alpha \}$ and $\mathcal{G}_\alpha = \{ \tau : E(\tau - \theta)^+ \leq \alpha \}$. The corresponding optimal stopping times are of the same form as above (with other $z^*$'s).

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1. A property of the moment of attaining the maximum. Let $B = (B_t)_{0 \leq t \leq 1}$ be a standard Brownian motion vanishing at zero and $\theta$ be the moment at which $B$ attains its maximal value, i.e., $B_\theta = \max_{0 \leq t \leq 1} B_t$ (such a random variable $\theta$ is defined a.s. in a unique way). Denote by $(\mathcal{F}^B_t)_{0 \leq t \leq 1}$ the filtration generated by $B$.

Lemma 1. For any $(\mathcal{F}^B_t)$-stopping time $\tau$ ($0 \leq \tau \leq 1$), the following equality holds:

$$E(B_\theta - B_\tau)^2 = E(\theta - \tau) + \frac{1}{2}. \hspace{1cm} (1)$$

Proof. 1) In the proof of the main result of the work [1] the authors establish that

$$E(B_\theta - B_\tau)^2 = 1 + E \int_0^\tau \left( 4\Phi \left( \frac{S_t - B_t}{\sqrt{1 - t}} \right) - 3 \right) dt. \hspace{1cm} (2)$$

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where \( S = (S_t)_{0 \leq t \leq 1} \) is the maximum process of \( B \), i.e., \( S_t = \max_{0 \leq s \leq 1} B_s \), and 
\( \Phi(x) \) is the distribution function of a standard Gaussian random variable.

For completeness, let us prove (2). We have

\[
E(B_0 - B_t)^2 = E(S_1 - B_t)^2 = E(S_t^2) - 2E(S_t B_t) + E(B_t^2).
\]

(3)

The random variable \( S_1 \) is known to possess the property \( S_1 = ES_1 + \int_0^1 H_t dB_t \),

where \( H_t = 2 - 2\Phi \left( \frac{S_t - B_t}{\sqrt{1 - t}} \right) \), \( 0 \leq t \leq 1 \) (see [4; p. 93]). Consequently,

\[
E(S_1 B_t) = E \left[ \left( \int_0^1 H_t dB_t \right) \left( \int_0^t dB_t \right) \right] = E \int_0^t H_t \, dt.
\]

(4)

Since \( E(B_t^2) = \sigma_t \) and \( E(S_t^2) = 1 \), (3) and (4) imply (2).

2) Define the process \( \pi_t = P(\theta \leq t | \mathcal{F}_t^B) \), \( 0 \leq t \leq 1 \). We have

\[
\pi_t = P(\theta \leq t | \mathcal{F}_t^B) = P(S_t \geq \max_{t \leq s \leq 1} B_s \mathcal{F}_t^B) = P(S_t - B_t \geq \max_{t \leq s \leq 1} (B_s - B_t) | \mathcal{F}_t^B).
\]

Since \( S_t - B_t \) is \( \mathcal{F}_t^B \)-measurable and \( \max_{t \leq s \leq 1} (B_s - B_t) \) is independent of the \( \sigma \)-field \( \mathcal{F}_t^B \), then the latter conditional probability equals the value of the distribution function of random variable \( \max_{t \leq s \leq 1} (B_s - B_t) \) at the point \( S_t - B_t \). As a consequence,

\[
\pi_t = 2\Phi \left( \frac{S_t - B_t}{\sqrt{1 - t}} \right) - 1, \quad 0 \leq t \leq 1.
\]

(5)

We have

\[
|\theta - \tau| = (\tau - \theta)^+ + \tau - \theta = \theta + \int_0^\tau (I(\theta \leq t) - I(\theta > t)) \, dt = \theta + \int_0^\tau (2I(\theta \leq t) - 1) \, dt.
\]

This implies

\[
E|\theta - \tau| = \frac{1}{2} + E \int_0^\tau (2I(\theta \leq t) - 1) \, dt = \frac{1}{2} + E \int_0^\tau (2I(\theta \leq t) - 1)I(\tau > t) \, dt
\]

\[
= \frac{1}{2} + E \int_0^\tau E(2I(\theta \leq t) - 1 | \mathcal{F}_t^B)I(\tau > t) \, dt = \frac{1}{2} + E \int_0^\tau (2\pi_t - 1) \, dt.
\]

Using (5), we get

\[
E|\theta - \tau| = \frac{1}{2} + E \int_0^\tau \left( 4\Phi \left( \frac{S_t - B_t}{\sqrt{1 - t}} \right) - 3 \right) \, dt.
\]

(6)

The formulas (2) and (6) finish the proof. \( \square \)

*Remarks.* (i) Denote by \( L = (L_t)_{0 \leq t \leq 1} \) the local time of a Brownian motion \( B \) at zero and put \( g = \sup \{0 \leq t \leq 1 : B_t = 0\} \). Then for any \( \mathcal{F}_t^B \)-stopping time \( \tau \) \( (0 \leq \tau \leq 1) \),

\[
E(L_g - L_\tau + |B_\tau|)^2 = E(g - \tau)^2 + \frac{1}{2}.
\]

Indeed, the processes \( B \) and \( L - |B| \) are known to have the same distribution; the filtrations generated by \( |B| \) and \( L - |B| \) coincide (see [3; Ch. VI, (2.2)]). It remains to apply Lemma 1.

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(ii) The random variable $\theta$ is not an $(\mathcal{F}_t^B)$-stopping time. The question arises whether the statement analogous to (1) is valid if we consider two stopping times.

Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion vanishing at zero. Then for any $(\mathcal{F}_t^B)$-stopping times $\sigma$ and $\tau$ such that $E\sigma < \infty$ and $E\tau < \infty$,

$$E(B_\sigma - B_\tau)^2 = E|\sigma - \tau|.$$  

Indeed,

$$E(B_\sigma - B_\tau)^2 = E(B_\sigma^2) + E(B_\tau^2) - 2E\left[\left(\int_0^\sigma dB_t\right)\left(\int_0^\tau dB_t\right)\right] = E\sigma + E\tau - 2E(\sigma \wedge \tau) = E|\sigma - \tau|.$$  

2. Stopping Brownian motion as close as possible to $\theta$. Here we try to find an $(\mathcal{F}_t^B)$-stopping time $\tau$, which is the most close to $\theta$ in some sense (or a stopping time $\tau$, such that $B_\tau$ is the most close to $B_0 = \max_{0 \leq t \leq 1} B_t$ in some sense). If $B$ describes evolution of stock prices on some time interval, then the financial motivation of such a problem is the following: we observe stock prices and we want to sell our stocks at the “optimal” time $\tau$ (the best is to sell our stocks at time $\theta$ but we cannot do so because $\theta$ is not a stopping time).

The first result in this direction was established by S.E. Graversen, G. Peskir, and A.N. Shiryaev (see [1]). The authors solved the problem

$$\inf_{\tau} E\left(\max_{0 \leq t \leq 1} B_t - B_\tau\right)^2,$$

where the infimum is taken over all $(\mathcal{F}_t^B)$-stopping times $\tau$ ($0 \leq \tau \leq 1$). Further results may be found in [2; Ch. VIII] and in [6].

Consider the optimal stopping problem

$$\inf_{\tau} E|\tau - \theta|.$$  

(8)

Due to Lemma 1, optimal stopping times in (7) and (8) are the same. Thus, by the results of [1], we immediately obtain the following statement.

**Theorem 2.** There exists a unique optimal stopping time $\tau_*$ in (8), and it has the form

$$\tau_* = \inf\{0 \leq t \leq 1: S_t - B_t \geq z_\star \sqrt{1-t}\},$$

where $S_t = \max_{0 \leq s \leq t} B_s$, $z_\star$ is a unique positive root of the equation $4\Phi(z) - 2\varphi(z) - 3 = 0$, $\varphi(z)$ and $\Phi(z)$ are the density and the distribution function of a standard Gaussian random variable.

Remark. (i) Consider the problem (8) on the time interval $[0, T]$. Then the optimal stopping time has the form

$$\tau_* = \inf\{0 \leq t \leq T: S_t - B_t \geq z_\star \sqrt{T-t}\}$$

with the same $z_\star$. It follows from the previous Theorem and self-similarity of Brownian motion.
(ii) Stock prices are positive, and Brownian motion may take negative values. That is why if we think about the financial motivation of (8), then it is natural to solve analogous problems for positive processes X instead of B (for instance, we may consider \( X = e^B \)). Since only times are involved in (8) (rather than values of the processes), then for any continuous strictly increasing function \( f \), the solution of the corresponding problem for \( X = f(B) \) follows from Theorem 2.

3. Risk minimization in some classes of stopping times. Here we solve two following problems:

\[
\inf_{\tau \in \mathcal{M}_a} E(\tau - \theta)^+ \tag{9}
\]

and

\[
\inf_{\tau \in \mathcal{N}_a} E(\tau - \theta)^-, \tag{10}
\]

where the classes of (\( \mathcal{F}_t^B \))-stopping times \( \mathcal{M}_a \) and \( \mathcal{N}_a \) are defined by the formulas \( \mathcal{M}_a = \{ 0 \leq \tau \leq 1: E(\tau - \theta)^- \leq \alpha \} \) and \( \mathcal{N}_a = \{ 0 \leq \tau \leq 1: E(\tau - \theta)^+ \leq \alpha \} \), \( 0 < \alpha < \frac{1}{2} \) (we use the notations \( x^+ = x \vee 0 \) and \( x^- = -(x \wedge 0) \)).

The financial motivation of these problems is similar to that of the previous section. We try to find a stopping time which is as close to \( \theta \) as possible, where the “closeness” is understood as follows. The values of \( (\tau - \theta)^- \) and \( (\tau - \theta)^+ \) are considered as the penalties for stopping prematurely \( (\tau < \theta) \) and for stopping too late \( (\tau > \theta) \). So, the problem (9) means that we minimize the expected penalty for stopping too late in the class of stopping times for which the expected penalty for stopping prematurely is not greater than \( \alpha \). The interpretation of (10) is analogous.

**Theorem 3.** For any \( \alpha \in (0, \frac{1}{2}) \), there exists a unique optimal stopping time \( \tau_* \) in (9), and it has the form

\[
\tau_* = \inf \{ 0 \leq t \leq 1: S_t - B_t \geq z, \sqrt{1-t} \}, \tag{11}
\]

where \( S_t = \max_{0 \leq s \leq t} B_s \) and a positive number \( z_* \) is uniquely determined by the condition

\[
E(\tau_* - \theta)^- = \alpha. \tag{12}
\]

(Also see Remark (i) at the end of the paper.)

**Theorem 4.** For any \( \alpha \in (0, \frac{1}{2}) \), there exists a unique optimal stopping time \( \tau_* \) in (10), and it has the form (11), where a positive number \( z_* \) is uniquely determined by the condition

\[
E(\tau_* - \theta)^+ = \alpha. \tag{13}
\]

(Also see Remark (i) at the end of the paper.)

We use the “Lagrange multiplier method” to solve (9) and (10). For any \( c > 0 \), we solve the problem

\[
\inf_{\tau} E[(\tau - \theta)^- + c(\tau - \theta)^+]. \tag{14}
\]

where the infimum is taken over all (\( \mathcal{F}_t^B \))-stopping times \( \tau \) \( (0 \leq \tau \leq 1) \). It is done in next Lemma. Then we prove Theorem 3. The proof of Theorem 4 is analogous.
**Lemma 5.** For any $c > 0$, there exists a unique optimal stopping time $\tau_*$ in the problem (14), and it has the form

$$\tau_* = \inf\{0 \leq t \leq 1: S_t - B_t \geq z_c \sqrt{1-t}\},$$

where $z_c$ is a unique positive root of the equation

$$(2c + 2)\Phi(z) - (c + 1)z\varphi(z) - (c + 2) = 0,$$  

(15)

$\varphi(z)$ and $\Phi(z)$ are the density and the distribution function of a standard Gaussian random variable.

**Proof.** 1) Analogously to the reasoning of part 2) of Lemma 1 we get

$$E[(\tau - \theta)^- + c(\tau - \theta)^+] = \frac{1}{2} + E \int_0^\tau F \left( \frac{S_t - B_t}{\sqrt{1-t}} \right) dt,$$

where

$$F(z) = (2c + 2)\Phi(z) - (c + 2).$$

The processes $S - B$ and $|B|$ are known to have the same distribution; the filtrations generated by $B$ and $S - B$ coincide (see [3, Ch. VI, (2.3), (2.12)]). Hence, (14) turns into the problem

$$\inf_{\tau} E \int_0^\tau F \left( \frac{|B_t|}{\sqrt{1-t}} \right) dt,$$  

(16)

where the infimum is taken over $(\mathcal{F}_t^{[B]})$-stopping times $\tau$ ($0 \leq \tau \leq 1$). Now we solve (16), where the infimum is taken over $(\mathcal{F}_t^B)$-stopping times. The optimal stopping time will turn out to be an $(\mathcal{F}_t^{[B]})$-stopping time. So, the proof will be finished.

Consider a deterministic time-change $\rho_s = 1 - e^{-2s}$, $s \geq 0$, which maps $\mathbb{R}_+$ onto $[0, 1)$. Put $Z_s = \frac{B_{\rho_s}}{\sqrt{\rho_s}} = e^s B_{\rho_s}$, $s \geq 0$. It is obvious that $\mathcal{F}_s^Z = \mathcal{F}_s^B$, $s \geq 0$. By Ito’s formula, the process $Z$ satisfies the equation

$$dZ_s = Z_s ds + \sqrt{2} d\beta_s,$$  

(17)

where $\beta_s = \int_0^s \frac{1}{\sqrt{1-t}} e^{u} dB_u$, $s \geq 0$. The process $\beta$ is an $(\mathcal{F}_s^Z)$-Brownian motion because it is a continuous local martingale vanishing at zero and $\langle \beta \rangle_s = s$.

It is obvious that $\sigma$ is an $(\mathcal{F}_s^Z)$-stopping time ($0 \leq \sigma \leq \infty$) if and only if $\rho_{\sigma}$ is an $(\mathcal{F}_s^B)$-stopping time ($0 \leq \rho_{\sigma} \leq 1$). Since

$$\int_0^{\rho_{\sigma}} F \left( \frac{|B_t|}{\sqrt{1-t}} \right) dt = 2 \int_0^\sigma e^{-2s} F(|Z_s|) ds,$$

then (16) turns into the problem

$$\inf_{\sigma} E \int_0^\sigma e^{-2s} F(|Z_s|) ds,$$  

(18)

where the infimum is taken over $(\mathcal{F}_s^Z)$-stopping times $\sigma$ ($0 \leq \sigma \leq \infty$).
2) Consider a family of measures \((P_z)_{z \in \mathbb{R}}\) such that under \(P_z\) the process \(Z\) satisfies (17) with the initial condition \(Z_0 = z\). Put

\[
V_*(z) = \inf_{\sigma} E_z \int_0^\sigma e^{-2s} F(|Z_s|) \, ds
\]

(19)

\(V_*(0)\) corresponds to (18), which is what we need to solve. It is natural to expect that there is an optimal stopping time in (19) of the form

\[
\sigma_* = \inf\{s \geq 0 : |Z_s| \geq z_\epsilon\},
\]

(20)

where \(z_\epsilon\) is some positive threshold to find. Note that \(|Z_s| \to \infty\) as \(s \to \infty\). Hence, \(\sigma_* < \infty \) \(P_z\)-a.s. In order to compute the value function \(V_*(z)\) and the threshold \(z_\epsilon\) let us formulate the following Stephan problem (see [5; Ch. III. § 8]):

\[
(L_Z - 2)V(z) = -F(|z|), \quad |z| < z_\epsilon,
\]

(21)

\[
V(\pm z_\epsilon) = 0,
\]

(22)

\[
V'(\pm z_\epsilon) = 0,
\]

(23)

where \(L_Z\) is the generator of \(Z\). It follows from (17) that \(L_Z = \frac{d^2}{dz^2} + z \frac{d}{dz}\). The formulas (19) and (17) imply that the function \(V_*(z)\) is even. That is why we solve the problem (instead of (21)-(23)):

\[
V''(z) + zV'(z) - 2V(z) = (c + 2) - (2c + 2) \Phi(z), \quad z \in (0, z_\epsilon),
\]

(24)

where the boundary conditions are

\[
V(z_\epsilon) = V'(z_\epsilon) = V'(0) = 0.
\]

General solution of the equation (24) is given by the formula

\[
V(z) = C_1(1 + z^2) + C_2[z\Phi(z) + (1 + z^2) \Phi'(z)] + (c + 1) \Phi(z) - \frac{c + 2}{2},
\]

where \(C_1\) and \(C_2\) are arbitrary constants. The condition \(V'(0) = 0\) implies that \(C_2 = -\frac{c + 1}{2}\). The condition \(V'(z_\epsilon) = 0\) implies that \(C_1 = \frac{c + 1}{2} \Phi(z_\epsilon)\). It follows from \(V(z_\epsilon) = 0\) that \(z_\epsilon\) should satisfy the equation (15). Finally, it is easy to check that for any \(c > 0\), the equation (15) has a unique positive root.

3) Let \(z_\epsilon\) be the unique positive root of (15). Put

\[
V(z) = \begin{cases} 
\frac{c + 1}{2} \Phi(z_\epsilon)(1 + z^2) - \frac{c + 1}{2}[z\Phi(z) + (1 + z^2) \Phi'(z)] \\
+ (c + 1) \Phi(z) - \frac{c + 2}{2} \\
0
\end{cases} \quad \text{if } 0 \leq z \leq z_\epsilon;
\]

\[
\text{if } z > z_\epsilon.
\]

Define function \(V(z)\) on \(\mathbb{R}\) as even function. Now let us prove that such a function \(V(z)\) coincides with the value function \(V_*(z)\) in the problem (19) and that the stopping time \(\sigma_*\) defined in (20) is optimal.

Clearly, \(V \in C^4(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{-z_\epsilon, z_\epsilon\})\). By Itô’s formula (under the measure \(P_z\)),

\[
e^{-2z} V(Z_s) = V(z) + \int_0^s e^{-2u} (L_Z - 2)V(Z_u) \, du + M_s,
\]

where \(M_s = \int_0^s e^{-2u} \, dW_u\).
where $M_s = \int_0^s e^{-2u}V'(Z_u)\sqrt{\sum}d\beta_u$ is an $(\mathcal{F}_t^Z)$-local martingale. Since the function $V'(z)$ is bounded, then $|M|_{\infty} \leq const < \infty$. Burkholder-Davis-Gundy inequalities yield that $M$ is a uniformly integrable martingale, whence $\mathbb{E}M_\infty = 0$ for any $(\mathcal{F}_t^Z)$-stopping time $\sigma$ $(0 \leq \sigma \leq \infty)$. Consequently, for any $(\mathcal{F}_t^Z)$-stopping time $\sigma$,

$$V(z) = E_z[e^{-2\sigma V(Z_\sigma)}] = E_z \int_0^\sigma e^{-2u} (L_Z - 2) V(Z_u) du \leq E_z \int_0^\sigma e^{-2u} F(|Z_u|) du.$$  

(25)

Indeed, it is easy to establish that $V(z) < 0$ for $z \in (-z_c, z_c)$ and $(L_Z - 2) V(z) \geq -F(|z|)$ for $z \in \mathbb{R} \setminus \{-z_c, z_c\}$. In addition, the stopping time $\sigma_*$ defined in (20) is a unique stopping time for which the equality holds in (25).

Thus, the function $V(z)$ coincides with the value function $V_*(z)$ in the problem (19) and $\sigma_*$ is a unique optimal stopping time. Returning to the initial setting (14), we obtain that the proof is finished.

Remark 3. It is easy to check that $z_c$ as a function of $c \in (0, \infty)$ is strictly decreasing, continuous, $\lim_{c \to 0} z_c = \infty$, and $\lim_{c \to \infty} z_c = 0$.

Proof of the Theorem. Put

$$\tau(z) = \inf\{0 \leq t \leq 1: S_t - B_t \geq z\sqrt{1-t}\}, \quad f(z) = \mathbb{E}(\tau(z) - \theta)^-.$$

It is easy to verify that the function $f(z)$, $z \in (0, \infty)$, is non-increasing, continuous, $\lim_{z \to 0} f(z) = \frac{1}{2}$, and $\lim_{z \to \infty} f(z) = 0$. Hence, there exists $z_* > 0$ such that $f(z)_* = 0$. Due to the properties of $z_c$ as a function of $c \in (0, \infty)$, there exists $c_* > 0$ such that $z_c = z_{c_*}$. Now Lemma 5 implies that $\tau(z_*)$ is a unique optimal stopping time in the problem (9).

If there exists $z_{**} > 0$ such that $f(z_{**}) = 0$ and $z_\bullet \neq z_{**}$, then $\tau(z_{**})$ is a unique optimal stopping time in (9). It contradicts the fact that $\tau(z_\bullet)$ is a unique optimal stopping time in this problem. Therefore, the condition (12) determines uniquely the positive number $z_\bullet$.

Remark 4. In order to find optimal stopping times in (9) and (10), we need to solve the equations (12) and (13) with respect to $z$. That is why it is interesting to find a deterministic algorithm for computing the functions

$$f(z) = \mathbb{E}(\tau(z) - \theta)^- \quad \text{and} \quad g(z) = \mathbb{E}(\tau(z) - \theta)^+,$$

where $\tau(z) = \inf\{0 \leq t \leq 1: S_t - B_t \geq z\sqrt{1-t}\}$.

It is proved in [6] that for any $\lambda > 0$, the equation

$$2(\lambda + 1)u \left( \Phi(u) - \frac{1}{2} \right) - [(\lambda + 1)u^2 + 1] \varphi(u) - u = 0$$

has a unique positive root $u_\lambda$. Further it is proved there that $u_\lambda$ as a function of $\lambda \in (0, \infty)$ is strictly decreasing, continuous, $\lim_{\lambda \to 0} u_\lambda = \infty$, and $\lim_{\lambda \to \infty} u_\lambda = 0$. Again using the results of [6] it can be established that for any $z > 0$,

$$g(z) = \frac{1}{\lambda} \left[ \frac{-\varphi(z)}{z} + \lambda \left( \Phi(z) - \frac{1}{2} \right) - \frac{2 - 2\Phi(z)}{1 + z^2} \right].$$
where $\lambda > 0$ satisfies $u_\lambda = z$. Finally, it follows from the proof of Lemma 5 that for any $z > 0$,

$$f(z) = (c + 1)\Phi(z) - \frac{c}{2} - 1 - cg(z),$$

where $c > 0$ satisfies $z_c = z$.

(ii) Consider “boundary” cases $\alpha = 0$ and $\alpha = \frac{1}{2}$ in the problems (9) and (10). It is easy to establish that for $\alpha = 0$, $\tau \equiv 1$ is a unique optimal stopping time in (9). And for $\alpha = \frac{1}{2}$, $\tau \equiv 0$ is a unique optimal stopping time in (9). Analogous statements are valid for the problem (10).

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References


